Klosin

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PS1

Deformations of Saito-Kurokawa Type (part 1):

joint w/ T. Berger.

Assumptions: k even, $k \geq 2$, p prime, $p > 2k-2$

$E/\mathbb{Q}_p$ finite, $\mathcal{O} \subset E$ r.o.i., ϖ = unif, $\mathbb{F} = \mathcal{O}/\varpi$

$\Sigma = \{p\}$.

χ mod p cycl. char. $= \bar{\varepsilon}$.

$\rho : G_\Sigma \to GL_2(\mathbb{F})$ irred, odd, $\rho = \bar{\rho}_f$, $f \in S_{2k-2}(1)$.

Prop. (Berger-K.): If there exists a crystalline irred.

$$\sigma : G_\Sigma \to GL_4(E) \text{ s.t. } \bar{\sigma}^{ss} = \chi^{k-2} \oplus \chi^{k-1} \oplus \rho,$$

then there exists G_Σ-stable lattice in E^4 s.t.

$$\bar{\sigma} \simeq \begin{pmatrix} \chi^{k-2} & a & b \\ & \rho & c \\ & & \chi^{k-1} \end{pmatrix}$$

with a, c non-trivial classes in $H^1_f(\mathbb{Q}, \rho(1-k))$.

Furthermore, if $\dim_{\mathbb{F}} H^1_f(\mathbb{Q}, \rho(1-k)) = 1$, then the isom class of $\bar{\sigma}$ is well-defined.

Pf. Existence: Ribet-style arg.

Uniqueness: we use iterated Fontaine-L. exts. point is b entry is an $H^1(\mathbb{Q}, \mathbb{F}(-1))$-torsor. $\boxtimes$

$H^1(\mathbb{Q}, \mathbb{F}(-1)) = 0$

Remarks: Why study such σ?

1) If $F \in S_k^{(2)}(1)$, and $F \equiv SK(f) \pmod{\varpi}$, then σ_F is such if F is not SK-type.

2) Paramodularity conjecture $(k=2, \Sigma \neq \{p\})$ - part 2.

<u>Thm (Berger-K.):</u> Let $k>2$. Let σ be as in the Prop., $\sigma \simeq \sigma^\vee \otimes \varepsilon^{2k-3}$. Assume $(i=1,2)$

$(*_i)$ $\dim_{\mathbb{F}} H^1_f(\mathbb{Q}, \rho(i-k)) = 1$.

$R_\rho \simeq \mathcal{O}$ (essentially p is not a congruence prime for f) and some congruence condition. Then σ is modular, i.e., $\exists\, F \in S_k^{(2)}(1)$ s.t. $\sigma_F \simeq \sigma$.

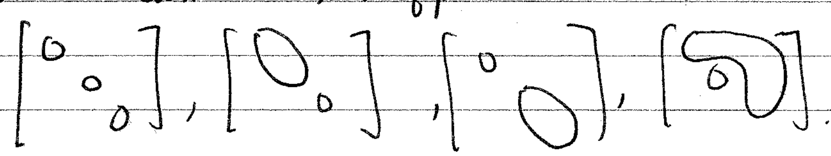
<u>Remark:</u> Note that we don't require $\bar\sigma$ to be modular.

<u>Sketch of proof:</u> $R' =$ univ. def. ring of $\bar\sigma$.

$$\tau : R'[G_\Sigma] \to R'[G_\Sigma] \qquad \tau(g) = \varepsilon^{2k-3}(g)\, g^{-1}.$$

$R =$ reduced q.t. of R' corresp. to deformations of γ s.t. $\operatorname{tr} \gamma = \operatorname{tr} \gamma \circ \tau$.

Let γ be such a deformation. There exist 4 ways in which $\operatorname{tr} \gamma$ can be the sum of pseudo characters:



$\rightsquigarrow$ we get 4 ideals $I \subset R$ s.t. R/I captures

Klosin

8-19-16

pg 3

such Y's. We show that $(*_2)$ implies the I's are equal and principal.

We then study deformations $Y = \begin{bmatrix} \varepsilon^{k-2} & a & b \\ & \rho_f & c \\ & & \varepsilon^{k-1} \end{bmatrix}$ (Urban)

$(*_1) \Rightarrow$ there are no infinitesimal such. $\Rightarrow R/_I \cong \mathcal{O}/_{\varpi^n}$.
(including possibility of infinite.)

Get

$$\# R/_I \leq \# \underbrace{H_f^1(\mathbb{Q}, \rho_f(1-k) \otimes E/\mathcal{O})}_{\text{finite}} \underset{\substack{\uparrow \\ \text{Bloch-Kato (Kato) thm}}}{\leq} \# \mathcal{O}/_{L_{alg}(k,f)}$$

$$\underset{\substack{\uparrow \\ \text{Sk-congruences (Brown)}}}{\leq} \# \mathbb{T}/_J \underset{\substack{\uparrow \\ R \twoheadrightarrow \mathbb{T} \\ \text{(uniqueness of } \bar{\sigma})}}{\leq} \# R/_I$$

$$\begin{array}{ccc} R & \twoheadrightarrow & \mathbb{T} \\ \downarrow & & \downarrow \\ \boxed{R/_{I}} & \xrightarrow{\sim} & \mathbb{T}/_J \end{array}$$

$\uparrow$ cyclic; I $\uparrow$ principal

$\Big\}$ commutative alg $\Rightarrow R \simeq \mathbb{T}$.

$\blacksquare$

Klosin
8-19-16
pg 4

<u>Remark</u>: If $k=2$, we don't have Saito-Kurokawa congruences. Plan is to ~~construct~~ deduce the missing congruence in wt 2 from a congruence b/w Hida families.