Herzig

8-15-16

PSI

General Serre weight conjectures

joint work with T. Gee, D. Savitt.

§1 Intro.

Serre's Conj: Let $\bar\rho : \Gamma_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ irred, odd. Then $\bar\rho$ arises from a modular form of level $\Gamma_1(\underbrace{N^?(\bar\rho)}_{\substack{\text{prime to } p \text{ and} \\ \text{only depends on } \{\bar\rho|_{I_\ell}\ (\ell \neq p)\}}})$ and weight $\underbrace{k^?(\bar\rho)}_{\substack{\text{depends only on} \\ \bar\rho|_{I_p}}} (\geq 2)$.

Weight part: If $\bar\rho$ arises from some eigenform, what are the possible weights (in prime to p level)?

Serre weights and unitary groups: F/F^+ CM extension

$$\begin{array}{ccc} F & & v \quad \bar v \\ | & & \backslash / \\ F^+ & & p \\ | & & | \\ \mathbb{Q} & & p \end{array}$$

G/F^+ compact unitary group. $G(F^+_w) \simeq U(n) \quad \forall w | \infty$.

$G(F_p^+) \cong GL_n(F_p^+)$

$S(U, M)$ — U: prime to p level; M: f.d. $\overline{\mathbb{F}}_p$-rep of $GL_n(\mathcal{O}_{F_p^+})$.

$\circlearrowleft$ space of mod p auto forms.

$\mathbb{T}$ Hecke alg. (outside p).

$\bar{r} : \Gamma_F \longrightarrow GL_n(\bar{\mathbb{F}}_p)$ irred. $\Rightarrow m_{\bar{r}} \triangleleft \mathbb{T}$ max ideal

$W(\bar{r}) := \{ \text{irred. } M : S(U, M)_{m_{\bar{r}}} \neq 0 \}$ "Serre weights"

"weights of $\bar{r}$" (some U prime to p)

$W(\bar{r})$ is always a finite set.

Assume $W(\bar{r}) \neq \phi$. (ie., $\bar{r}$ comes from an auto form)

Q: Determine $W(\bar{r})$.

Conj. 1: $W(\bar{r})$ depends only on $\bar{r}|_{\Gamma_{F_v}}$, even only on $\bar{r}|_{I_{F_v}}$.

1^{st} part should follow from local-global compatibility in mod p Langlands.

From now on, consider $K/\mathbb{Q}_p$ finite, $\bar{\rho} : \Gamma_K \longrightarrow GL_n(\bar{\mathbb{F}}_p)$.
To simplify notation, assume $K = \mathbb{Q}_p$; residue field $k = \mathbb{F}_p$.

§2. Breuil-Mezard conj:

$\mathbb{Z}_+^n := \{ \lambda \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n \}$.

$\rightsquigarrow L_\lambda :=$ irred. alg. rep. of GL_n over $\bar{\mathbb{Q}}_p$ of h.w. λ.
(considered as rep. of $GL_n(\mathcal{O}_K)$)

$X_1^{(n)} := \{ a \in \mathbb{Z}^n : 0 \leq a_i - a_{i+1} \leq p-1 \ \forall i \} \subseteq \mathbb{Z}_+^n$.

$\rightsquigarrow$ $F_a :=$ irred. alg. rep. of GL_n over $\overline{\mathbb{F}}_p$ of h.w. a.
(considered as rep. via $GL_n(\mathcal{O}_K) \to GL_n(k)$)

Say $a \sim b \iff a - b \in (p-1, \ldots, p-1)\mathbb{Z}$.

Then $X_1^{(n)}/\sim \;\xrightarrow{\sim}\; \{$ Serre wts = irred (smth) $\overline{\mathbb{F}}_p$-reps of $GL_n(\mathcal{O}_K) \}/\simeq$

$$a \longmapsto F_a.$$

<u>Ex:</u> $n = 2$. $F_{(x,y)} = Sym^{x-y}(\overline{\mathbb{F}}_p^{\,2}) \otimes \det^y$ $\quad 0 \le x - y \le p-1$

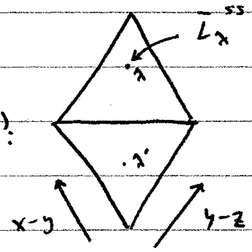
<u>Remark:</u> In general, $[\bar{L}_\lambda : F_\lambda] = 1$ $\quad (\lambda \in X_1^{(n)})$

but usually $\bar{L}_\lambda$ is much bigger.

<u>$n = 3$:</u>

$\bar{L}_\lambda^{ss} \simeq F_\lambda \oplus F_{\lambda'}$.

$X_1^{(3)}$:



$\lambda \in \mathbb{Z}_+^n$:

$R_{\bar{\rho}}^{\lambda}$:= Galois lifting ring parameterizing all lifts to $\rho : \Gamma_K \to GL_n(\overline{\mathbb{Z}}_p)$ crystalline of HT weights $(\lambda_1 + (n-1), \lambda_2 + (n-2), \ldots, \lambda_n)$, "type λ"

reduction $\bar{\rho}$.

Herzig

8-15-16

Conj: (BM $_{n=2}$, Emerton-Gee): For any Serre weight V there exists $! \, n_V(\bar\rho) \in \mathbb{Z}_{\geq 0}$ s.t. $\forall \lambda \in \mathbb{Z}_+^n$,

$$\underset{\substack{\uparrow \\ \text{HS mult}}}{e}\left(R_{\bar\rho}^{\lambda} \otimes \bar{\mathbb{F}}_p\right) = \sum_V [\bar L_\lambda : V]\, n_V(\bar\rho).$$

Conj. 2: $W(\bar\rho) = \{ V : n_V(\bar\rho) > 0 \}$

If these conjectures hold, then $\bar\rho$ has a crystalline lift of type λ iff $W(\bar\rho) \cap JH(\bar L_\lambda) \neq \emptyset$.

In particular, $F_a \in W(\bar\rho) \Rightarrow \bar\rho$ has a cryst. lift of type a.

Converse is false by recent work of Le Hung-Le-Levin-Morra.

§3 Explicit Conjectures:

In general, we expect at least that $W(\bar\rho) \subset W(\bar\rho^{ss})$.

From now on assume $\bar\rho$ is semisimple.

Conj. 3: If $\bar\rho$ is semisimple, then $W(\bar\rho) = \{ F_a : \bar\rho$ has a cryst. lift of type $a \}$.

Herzig
8-15-16
p5

$\Leftrightarrow \left(W(\bar\rho) \cap JH(\bar{L}_a) \neq \phi \underset{(**)}{\Rightarrow} F_a \in W(\bar\rho) \right).$

Our explicit conjecture

- obvious weights
- add more weights to satisfy $(**)$, ...

Obvious weights:

Ex: $n=3$ $\quad \bar\rho \simeq \begin{pmatrix} \omega^a & & \\ & \omega^b & \\ & & \omega^c \end{pmatrix}$ $\quad 1 < a-b,\ b-c,\ a-c < p-1.$

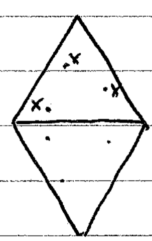
obv. cryst. lifts $\varepsilon^a \oplus \varepsilon^b \oplus \varepsilon^c \rightsquigarrow$ obv. wt. $F(a-2, b-1, c)$.

$\varepsilon^{a-(p-1)} \oplus \varepsilon^b \oplus \varepsilon^c \rightsquigarrow F(b-2, c-1, c-p+1)$

Get 6 obvious weights.

~~Shadow weights get from~~

Shadow weights: get from obvious wts using closure $(**)$.



In above example get three more Serre weights.

Define $W_{expl.}(\bar\rho)$. Generalizing earlier work have another set $W^2(\bar\rho)$.

<u>Thm</u>: If $\bar\rho$ is semisimple ($K/\mathbb{Q}_p$ unram) and some suitable genericity holds, then $W^2(\bar\rho) = W_{expl}(\bar\rho)$.