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## Cuspidal vs. Eisenstein cohomology:

Baby case: $\Gamma = GL_2(\mathbb{Z})$ Symmetric space $\mathbb{H}_+ \cup \mathbb{H}_- = X$

$= GL_2(\mathbb{R}) / SO(2)\,\mathbb{R}_{>0}$

$$\Gamma \backslash X = SL_2(\mathbb{Z}) \backslash \mathbb{H}_+$$

$\mathcal{M}_\lambda$ $\quad \lambda : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longrightarrow \left( t_1/t_2 \right)^{n/2} (t_1 t_2)^{d_0}$ $\quad n \geq 0$, $n$ even.

$n \equiv 2d_0 \pmod 2$.

character on maximal torus.

$\mathcal{M}_\lambda$ is a $\mathbb{Z}$-module w/ $\Gamma$-action.

$$\mathcal{M}_\lambda = \left\{ \sum a_\nu x^\nu y^{n-\nu} : a_\nu \in \mathbb{Z} \right\}$$

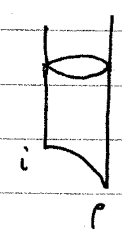
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \gamma P(x,y) = P(ax + cy,\ bx + dy) \det(\gamma)^{\frac{2n-d_0}{2}}.$$

Sheaf on $SL_2(\mathbb{Z}) \backslash \mathbb{H}_+ : \tilde{\mathcal{M}}_\lambda$

Consider the cohomology $H^1(SL_2(\mathbb{Z}) \backslash \mathbb{H}, \tilde{\mathcal{M}}_\lambda) = \mathcal{M}_\lambda / \mathcal{M}_\lambda^{(S)} + \mathcal{M}_\lambda^{(R)}$

where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

$$T_p : H^1(SL_2(\mathbb{Z}) \backslash \mathbb{H}, \tilde{\mathcal{M}}_\lambda) \circlearrowright$$




← boundary if cut off above a line.

$$H^1(\Gamma \backslash X, \mathcal{M}_\lambda) \xrightarrow{r} H^1(\partial(\Gamma \backslash \mathbb{H}), \mathcal{M}_\lambda)$$

$$H^1(\Gamma \backslash X, \mathcal{M}_\lambda)_{int} \longrightarrow H^1(\partial(\Gamma \backslash X), \mathcal{M}_\lambda)_{int} \longrightarrow 0$$

$\uparrow$ divide by the torsion $\qquad \downarrow$

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$$0 \to H_{int!} \longrightarrow H^1(\Gamma\backslash X, M_\lambda)_{int} \to H^1(\partial(\Gamma\backslash \mathbb{H}), M_\lambda)_{int} \to 0$$

$$\downarrow$$

$$H_{int!} \otimes \mathbb{Q} \longrightarrow H^1(\Gamma\backslash X, M_\lambda) \otimes \mathbb{Q} \to H^1(\partial(\Gamma\backslash \mathbb{H}), M_\lambda \otimes \mathbb{Q})$$

$$H^1(\partial(\Gamma\backslash X), M_\lambda)_{int} = \mathbb{Z}\,\omega_n \qquad T_p\,\omega_n = (p^{n+1}+1)\,\omega_n$$

$$H^1(\Gamma\backslash X, M_\lambda) \otimes \mathbb{Q} = H_{int!} \otimes \mathbb{Q} \oplus \mathbb{Q}\tilde{\omega}_n \qquad r(\tilde{\omega}_n) = \omega_n$$

$$T_p\,\tilde{\omega}_n = (p^{n+1}+1)\,\tilde{\omega}_n.$$

$\tilde{\omega}_n$ Eisenstein class, $\tilde{\omega}_n \notin H^1(\Gamma\backslash X, M_\lambda)_{int}$.

$$\Delta(n)\,\tilde{\omega}_n \in H^1(\Gamma\backslash X, \tilde{M}_\lambda)_{int}.$$

Thus, $H^1(\Gamma\backslash X, \tilde{M}_\lambda)_{int} \supset H^1_{int,!} \oplus \Delta(n)\,\tilde{\omega}_n$.

<u>"Thm"</u> $\Delta(n) = \mathrm{Num}(\zeta(-1-n))$.

This theorem is true for $n < 150$ by computer experiment. Then realized he had already proven the theorem 20 years ago, but doesn't trust the computations so needs to go through them again to claim it as a real theorem.

$$n = 10 \qquad H^1(\Gamma\backslash X, M_\lambda) = \mathbb{Z}f_0 + \overbrace{\mathbb{Z}f_1 + \mathbb{Z}f_2 + \mathbb{Z}f_3}^{H_{int!}}$$

↖ [illegible]

$$\tilde{\omega}_n = f_0 - \frac{2360}{691} f_1 \qquad T_2 = \begin{pmatrix} 2049 & -68040 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{pmatrix}$$

We now come to the main part of the talk. Replace

$\Gamma \backslash X$ by

$$S^G_{K_f} = G(\mathbb{Q}) \backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A})/K_f \qquad \dim d.$$

$$= \bigcup \Gamma_{g_f} \backslash X \qquad \underline{g_f} \in G(\mathbb{A}_f).$$

$\lambda$ highest weight, $\mathcal{M}_\lambda$ highest weight module.

$$H^q_c(S^G_{K_f}, \mathcal{M}_\lambda) \to H^q(S^G_{K_f}, \mathcal{M}_\lambda) \xrightarrow{r} H^q(\partial(S^G_{K_f}), \mathcal{M}_\lambda) \to H^{q+1}_c(\ldots)$$

$$H^q_c(S^G_{K_f}, \mathcal{M}_\lambda) \searrow H^q_!(S^G_{K_f}, \mathcal{M}_\lambda) \nearrow H^q(S^G_{K_f}, \mathcal{M}_\lambda)$$

$T^{(x)}_q$ acts upon these groups.

$$H^q(S^G_{K_f}, \mathcal{M}_\lambda) \to \mathrm{Im}(r)_q \subset H^q(\partial(S^G_{K_f}), \mathcal{M}_\lambda) \to$$

$$H^q(\partial(S^G_{K_f}), \mathcal{M}_\lambda)$$

$$\cup$$

$$\bigoplus_{\substack{P, w \in W^P \\ P \to M}} H^{q-\ell(w)}(S^M_{K^M_f}, M_{w\cdot\lambda})_{[\sigma_f]}.$$

$$\vdots$$

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Assume $S^G_{K_f}$ Shimura.

$\boxed{\text{General expectation:}}$

$$H^q(S^G_{K_f}, M_\lambda) \supset H^q_!(S^G_{K_f}, M_\lambda) \oplus \Delta(\sigma_f)\, \mathcal{Y}(\sigma_f).$$

$$\Delta(\sigma_f) \text{ ``=" } \mathrm{Denom}\left(\frac{1}{\Omega(\sigma_f)} \frac{\mathcal{L}(\sigma_f, n)}{\mathcal{L}(\sigma_f, n+1)}\right)$$

expression in terms of critical values

$\Omega(\sigma_f)$ period unique up to unit

This is explained on homepage in mix-mot. 2015.

$\Delta(n)$ = denom. $\Rightarrow$ congruences