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Weil-étale cohomology and Zeta functions of arithmetic schemes

joint w/ B. Morin.

X regular scheme proper over $\mathbb{Z}$.
$\zeta(s,X)$ the associated zeta function.
Interested in special value conjecture for $\zeta(s,X)$ at $s=n\in\mathbb{Z}$.

One normally encounters $L(s, h^i(X_{\mathbb{Q}}))$:

BSD $i=n=1$.

T.N.C. any i, n.

The point of view here was advocated by Lichtenbaum, Milne, Geisser, ..

Motivic cohomology:

motivic complex $\mathbb{Z}(n)$ on X_{et}.

$n\geq 0$ $\mathbb{Z}(n)$ Bloch's higher Chow complex. $(\mathbb{Z}(0)=\mathbb{Z})$

$n<0$ Define $\mathbb{Z}(n)$ via $Rf_*\mathbb{Z} \simeq \mathbb{Z}\oplus\mathbb{Z}(-1)[-2]\oplus\cdots\oplus\mathbb{Z}(-n)[-2n]$

where $f: \mathbb{P}^n_{X,et}\to X_{et}$. for sim.

Key Assumptions:

1) $H^i(X_{et},\mathbb{Z}(n))$ is a f.g. ab. group $i\leq 2n+1$.

Ex: $H^1((\operatorname{Spec}\mathbb{F}_q)_{et},\mathbb{Z}) = H^1(\underset{\hat{\mathbb{Z}}\,\supseteq\,\mathbb{Z}\cdot\mathrm{Frob}\,=\,W_{\mathbb{F}_q}}{G_{\mathbb{F}_q}},\mathbb{Z})$

$$= \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Q}/\mathbb{Z} & i=2. \end{cases}$$

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$$H^i(W_{\mathbb{F}_q}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & o/w \end{cases}.$$

Lichtenbaum showed this works for $X/\mathbb{F}_q$.

2) Artin-Verdier duality for $\mathbb{Z}(n)/m \leftrightarrow \mathbb{Z}(d-n)/m$

Known if $X \to \operatorname{Spec} \mathcal{O}_F$ smooth for example.
$\operatorname{Spec} \mathbb{F}_q$

m invertible on X, $\mathbb{Z}(n)/m \simeq \mu_m^{\otimes n}$

These assumptions allow one to define: a) perfect complex of abelian groups

$$R\Gamma_{c,w}(X, \mathbb{Z}(n)).$$

b) exact triangle in D^b(loc. compact. ab. groups).

$$(\mathbb{Z} \to \mathbb{R}) \simeq \mathbb{R}/\mathbb{Z}[0]$$

$\mathbb{Z} \oplus \mathbb{Z}\alpha \longrightarrow \mathbb{R}$ can't simplify this one.
$\alpha \notin \mathbb{Q}$.

(*) $R\Gamma_{c,w}(X, \mathbb{Z}(n)) \to R\Gamma_{c,w}(X, \mathbb{R}(n)) \to R\Gamma_{c,w}(X, \mathbb{R}/\mathbb{Z}(n))$.

1) $R\Gamma_{c,w}(X, \mathbb{R}(n))$ is a perfect complex of $\mathbb{R}$-v.s.

$\to H^i_{c,w}(X, \mathbb{R}(n)) \xrightarrow{\cup\theta} H^{i+1}_{w,c}(X, \mathbb{R}(n)) \xrightarrow{\cup\theta}$ l.e.s.

2) $H^i_{c,w}(X, \mathbb{R}/\mathbb{Z}(n))$ are compact Lie groups.

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Conj:

Relation to $\zeta(s,X)$ at $s=n \in \mathbb{Z}$ is given by

1) $\operatorname{ord}_{s=n} \zeta(s,X) = \sum (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_{W,c}(X, \mathbb{R}(n))$.

2) $\underset{\mathbb{R}}{\underset{\Cap}{|\zeta^*(n,X)|^{-1}}} = \prod_{i \in \mathbb{Z}} \operatorname{vol}(H^i_{W,c}(X, \mathbb{R}/\mathbb{Z}(n)))^{(-1)^i}$.

G loc. compact abelian group. "tangent space"

$$T_\infty G = \operatorname{Hom}_{cts}(\operatorname{Hom}_{cts}(G, \mathbb{R}/\mathbb{Z}), \mathbb{R})$$

T_∞ applied to (*) gives

$$R\Gamma(X_{Zar}, L\Omega^{\cdot}_{X/\mathbb{Z}}/F^n)_{\mathbb{R}}[-2] \to R\Gamma_{c,W}(X, \mathbb{R}(n))$$

derived de Rham complex (Illusie) $\quad \to R\Gamma_{c,W}(X, \mathbb{Z}(n))_{\mathbb{R}}$

Taking $\det_{\mathbb{R}}$ gives

$$\det_{\mathbb{R}} R\Gamma_{c,W}(X, \mathbb{Z}(n))_{\mathbb{R}} \simeq \det_{\mathbb{R}} R\Gamma(X_{Zar}, L\Omega^{\cdot}_{X/\mathbb{Z}}/F^n)[-1].$$

$$\cup \qquad\qquad \cup$$

$$\det_{\mathbb{Z}} R\Gamma_{c,W}(X, \mathbb{Z}(n)) = \lambda \cdot C(X,n) \det_{\mathbb{Z}}^{-1} R\Gamma(X_{Zar}, L\Omega^{\cdot}_{X/\mathbb{Z}}/F^n).$$

$C(X,n)$ correction form. $C(X,n) \in \mathbb{Q}^\times$

$\lambda \in \mathbb{R}_{>0}$.

$C(X, n) = 1$ if $n \leq 0$ or $X/\mathbb{F}_p$. (Morin)

$n \geq 1$

$$\zeta(n, \operatorname{Spec} \mathbb{F}_q) = (1 - q^{-n})^{-1} = \frac{q^n}{q^n - 1}$$

$q^n = \# L\Omega_{\mathbb{F}_q/\mathbb{Z}} / F^n$

$q^n - 1 = \# H^1(X, \mathbb{Z}(n)) = K_{2n-1}(\mathbb{F}_q)$

In general, $C(X, n) = \prod_p C_p(X, n)$

$C_p(X,n)$ defined via p-adic Hodge theory and forced by compatibility with TNC (Tamagawa number conjecture).

$X = \operatorname{Spec} \mathcal{O}_F$ $\quad F = \#$ field for the rest of the talk.

$$0 \to \prod_{v \mid \infty} F_v^\times / \mathcal{O}_F^\times \to H^2_{W,c}(X, \mathbb{Z}(1)) \to Cl(\mathcal{O}_F) \to 0$$

$$\uparrow \exp \qquad\qquad \downarrow$$

$$\prod_{v \mid \infty} F_v = F \otimes \mathbb{R} \qquad H^2_{W,c}(X, \mathbb{R}(1)) = \mathbb{R}$$

$H^3_{W,c}(X, \mathbb{R}(1)) = \mathbb{R}$

$$0 \to \underbrace{H^3_{W,c}(X, \mathbb{Z}(1))}_{= \mathbb{Z}} \to \underbrace{H^3_{W,c}(X, \mathbb{R}(1))}_{= \mathbb{R}} \to \mathbb{R}/\mathbb{Z} \to 0$$

$n \geq 2$: $H^2_{W,c}(X, \mathbb{Z}(n)) \simeq H^1_{W,c}(X, \mathbb{R}/\mathbb{Z}(n))$ compact

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Motivic coh. of Spec $\mathcal{O}_F$.

$$H^i(X_{et}, \mathbb{Z}(n)) \simeq H^i(X_{Zar}, \mathbb{Z}(n)) \underset{2}{\sim} K_{2n-i}(\mathcal{O}_F) \qquad i = 1, 2.$$

$$h_n = |H^2(X_{Zar}, \mathbb{Z}(n))|, \quad w_n = |H^1(X_{Zar}, \mathbb{Z}(n))_{tor}|.$$

R_n = covolume of r_n (Bloch-Goncharov regulator)

$$r_n : H^1(X, \mathbb{Z}(n)) \to H^1_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)) = \prod_{v|\infty} H^0(F_v, (2\pi i)^{n-1}\mathbb{R})$$

For $n \geq 2$ our conjecture is equivalent to

$$(**) \quad \zeta_F(n) = \underbrace{(n-1)!^{-[F:\mathbb{Q}]}}_{C(X,n)} |D_F|^{1-n} \frac{2^{r_1(-1)^{n-1}} (2\pi)^{n[F:\mathbb{Q}] - r_2 - r_1 \frac{1-(-1)^n}{2}} h_n R_n}{w_n \sqrt{|D_F|}}$$

derived deRham

only known for $F_v/\mathbb{Q}_p$ abelian.

In general, $C(X,n)$ can be guessed from functional equation since $C(X, d-n) = 1$.

Thm: $(**)$ holds for $n = 0, 1$ and $F/\mathbb{Q}$ abelian for any $n \in \mathbb{Z}$.

additive analogue of s.s. from $H^1(X, \mathbb{Z}(n)) \Rightarrow K(X)$

$$H^{2n-i}_{dR}(L\Omega^{\cdot}_{X/\mathbb{Z}} / F^n) \Rightarrow HC^L_i(\mathcal{O}_F/\mathbb{Z})$$

$X = \mathcal{O}_F$

$$HC^L_i(\mathcal{O}_F/\mathbb{Z})_{\mathbb{Q}} \simeq HC^L_i(F/\mathbb{Q}) \simeq \mathrm{Prim}\, H^{Lie}_i(M_\infty(F), \mathbb{Q})$$

possible source for $C(X,n)$