Geometric Serre weights for Hilbert modular forms:

joint work w/ S. Sasaki (work in progress)

Recall Serre's Conjecture $/\mathbb{Q}$:

Thm: If $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is continuous odd and irreducible, then ρ is modular of level $N(\rho)$ and weight $k(\rho)$.

Modularity is due to Khare-Wintenberger.
Level is due to Ribet, et. al.
Weight is due to Gross, et. al.

ρ is modular if $\rho \simeq \bar{\rho}_f$ for some Hecke eigenform

$f \in S_{k(\rho)}(\Gamma_1(N(\rho)))$.

$N(\rho)$ = Artin conductor of ρ (prime to p)

$k(\rho)$ is the least $k \geq 2$ s.t. $\rho|_{G_{\mathbb{Q}_p}}$ has a crystalline lift w/ H-T weights $(0, k-1)$.

Extends to a version (Edixhoven) that allows $k=1$:
Any ρ is geometrically modular of level N and weight k if $\rho \simeq \rho_f$ for some Hecke eigenform $f \in H^0(X_1(N)_{\overline{\mathbb{F}}_p}, \omega_k)$.
(= mod p modular forms of wt k, level N).

$k_{geom}(\rho)$ = least $k \geq 1$ s.t. $\rho|_{G_{\mathbb{Q}_p}}$ has a crystalline lift w/ H-T weight $(0, k-1)$.

In particular, $K_{geom}(\rho) = 1$ iff H-T weights $(0,0)$,
i.e., ρ is unramified at p $\Leftrightarrow$ ρ is geometrically modular of weight 1

Now suppose F is totally real and consider $\rho: G_F \to GL_2(\overline{\mathbb{F}}_p)$ continuous, irred., and totally odd.

Generalization of Serre's conjecture predicts that $\rho \cong \overline{\rho}_f$ for some Hilbert modular ~~form~~ eigenform f, some weight and level.

Known that prime-to-p part of the level can be taken to $N(\rho)$ (Artin conductor).
Weight? $\in \mathbb{Z}^I \times \mathbb{Z}$, $I = \{F \hookrightarrow \overline{\mathbb{Q}}\}$ $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$
(n_τ, w) "paritious" i.e. $n_\tau \equiv w \pmod 2$

Let V be an irreducible representation of $GL_2(\mathcal{O}_F/p)$ over $\overline{\mathbb{F}}_p$, so $V = \bigotimes_{\tau: \mathcal{O}_F \to \overline{\mathbb{F}}_p} (\det^{m_\tau} \otimes \mathrm{Sym}^{k_\tau - 2} \overline{\mathbb{F}}_p^2) = \bigotimes_{\mathfrak{p}|p} V_{\mathfrak{p}}$.

$$\left(GL_2(\mathcal{O}_F/p) \twoheadrightarrow \prod_{\mathfrak{p}|p} GL_2(\mathcal{O}_F/\mathfrak{p}) \right)$$

with $2 \le k_\tau \le p+1$

Say ρ is cohomologically modular of weight V if ρ arises in $H^1_{\acute{e}t}(X_{U,\overline{F}}, \mathcal{F}_V)$ where X_U is a Shimura curve associated to a quaternion algebra D/F split at all $\mathfrak{p}|p$ and exactly one ∞-prime and level

$U \subseteq (D \otimes \mathbb{A}_F^\infty)^\times$ sufficiently small and prime to p. $\mathcal{F}_V$ is the sheaf associated to V.

__Thm (Gee, et al):__ If ρ is modular, $p>2$, and satisfies a Taylor-Wiles hypothesis, then ρ is cohomologically modular of weight V iff $\rho|_{G_{F_p}}$ has a crystalline lift with H-T weights corresponding to V_p.

eg if p is unramified, and $V_p \cong \bigotimes_{\tau: \mathcal{O}_{F/p} \to \overline{\mathbb{F}}_p} \mathrm{Sym}^{k_\tau - 2} \overline{\mathbb{F}}_p^2$ this means τ-labelled weights are $(0, k_\tau - 1)$.

This determines all cohomological weights $(\in \mathbb{Z}^I_{\geq 2} \times \mathbb{Z})$ and ~~[illegible]~~ level structures at p of Hilbert modular forms f giving rise to ρ.

What about "geometric modularity"?
For simplicity assume p is inert in F. (generalization to p unramified is straightforward) Assume $F \neq \mathbb{Q}$.
Let Y_U be the Hilbert modular variety$/_{\mathbb{Q}}$ of level $U \subseteq GL_2(\mathbb{A}_F^\infty)$, prime to p,

finite adeles

so $Y_U(\mathbb{C}) = GL_2(F) \backslash GL_2(\mathbb{A}_F^\infty) \times (\mathbb{C} \setminus \mathbb{R})^I / U$

and Y_U has good reduction $\overline{Y}_U/_{\overline{\mathbb{F}}_p}$.

For U suff. small; e.g. $U = U_1(\mathfrak{n})$ for suff. small $\mathfrak{n}$, in particular $x \in \mathcal{O}_F^\times$, $x \equiv 1 \pmod{\mathfrak{n}} \Rightarrow x \equiv 1 \pmod{p}$, can define automorphic line bundles ω_k on $\overline{Y}_U$ for any $k \in \mathbb{Z}^I$ so that $H^0(\overline{Y}_U, \omega_k)$ deserves to be called the space of mod p Hilbert modular forms of weight k and level U.

Hecke operators T_v, S_v for $v \nmid p\mathfrak{n}$. act on $H^0(\overline{Y}_U, \omega_k)$.

Say ρ is <u>geometrically modular</u> of weight k if $\exists$ eigenform $f \in H^0(\overline{Y}_U, \omega_k)$ s.t. $T_v f = tr(\rho(Frob_v)) f$ and $(N_{F/\mathbb{Q}} v) S_v f = det(\rho(Frob_v)) f$.

<u>Conjecture 1:</u> If $2 \leq k_\tau \leq p+1$, then ρ is geometrically modular of wt k iff ρ is cohomologically modular of wt $\bigotimes_{\tau : \mathcal{O}_F/p \to \overline{\mathbb{F}}_p} Sym^{k_\tau - 2} \overline{\mathbb{F}}_p^2$.

(p inert, so identify $I = \{F_p \hookrightarrow \overline{\mathbb{Q}}_p\}$

$= \{\mathcal{O}_F/p \hookrightarrow \overline{\mathbb{F}}_p\} = \{0, \ldots, d-1\}$.)

<u>Remarks:</u> Can formulate a version allowing any $k_\tau \in \mathbb{Z}^I_{\geq 2}$ and incorporating twisting.

If k is paritious, then $\Leftarrow$ is ok.

Diamond
8-19-16
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Ideally, want to characterize all geometric weights in terms of $\rho|_{G_{F_\mathfrak{p}}}$.

By partial Hasse invariants (a la Andreatta-Goren) $(0, \ldots, p, -1, 0, \ldots, 0)$, $\exists$ minimal geometric wt $K_{geom}(\rho)$ s.t. all weights are $K_{geom}(\rho) + \Sigma^+_{AG}$.

Question: do $K_{geom}(\rho) \in \Sigma^+$ or $\in \Sigma^+_{AG}$?

<u>Conjecture 2</u>: If $k \in \Sigma^+$, then ρ is geom. modular of wt k iff $\rho|_{G_{F_\mathfrak{p}}}$ has a cusp. lift of wt k corresponding k.

<u>Thm</u>: Suppose F is quadratic. If ρ is modular and satisfies Taylor-Wiles hypothesis and Conj. 1 holds, and $2 \le m \le p-1$, then ρ is modular of weight $(1, m)$ $\Leftrightarrow$ ρ has a crystalline lift of corresponding weight.