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Deformations of Saito-Kurokawa type (part 2):

1. Paramodularity conjecture (Brumer-Kramer):

$K(N)$ = paramodular group of level N.

$= \mathrm{diag}(1,1,N,1)\, \mathrm{Mat}_4(\mathbb{Z})\, \mathrm{diag}(1,1,N,1)^{-1} \cap Sp_4(\mathbb{Q})$

$$\left\{ \begin{array}{l} \text{abelian surfaces of } A_{/\mathbb{Q}} \text{ of} \\ \text{cond. } N,\ \mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z} \end{array} \right\}_{/\sim} \longleftrightarrow \left\{ \begin{array}{l} F \in S_2^{(2)}(K_0(N)) \text{ eigenforms} \\ \text{not in span of SK-lifts,} \\ \text{rational e.v.'s.} \end{array} \right\}_{/\mathbb{C}^\times}$$

s.t. $L(s, A) = L(s, F; spin)$

$\Rightarrow \sigma_{A,p} = V_p A \simeq \sigma_{F,p} : G_{\mathbb{Q}_\Sigma} \longrightarrow GSp_4(\mathbb{Q}_p)$.
for every prime $\qquad \Sigma = \{p, \ell \mid N\}$.

Saito-Kurokawa / Gritsenko:

$$S_2^-(\Gamma_0(N)) \longrightarrow S_2^{(2)}(K(N))$$
$$f \longmapsto SK(f)$$

$\rightsquigarrow$ Gal. rep.

$$\sigma_{SK(f)} = \mathbb{1} \oplus \rho_f \oplus \varepsilon$$

p-adic cycl.

2. Progress towards conj:

Let A be as in the conj. Assume N is prime, $p>2$, $p \nmid N$. Assume $A(\mathbb{Q})[p] \neq \{0\}$ and A has polarization of degree prime to p. Then

$$\bar{\sigma}_{A,p}^{ss} = \mathbb{1} \oplus \bar{\rho}^{ss} \oplus \underbrace{\chi}_{\text{mod } p \text{ cycl.}}$$

Assume $\bar{\rho}$ is irred. $\overset{\text{Serre conj}}{\Longrightarrow}$ $\bar{\rho} = \bar{\rho}_f$ for f wt 2 level N.

Assume root # $\varepsilon_f = -1$. Then $\bar{\sigma}_{A,p}^{ss} \simeq \bar{\sigma}_{SK(f)}$.

"Thm" Assume A has semi-stable reduction at N

- $\dim_{\mathbb{F}} H^1_f(\mathbb{Q}, \bar{\rho}_f) = 1$, $R_{\bar{\rho}_f} = \mathcal{O}$, $\bar{\rho}_f|_{I_N} \neq 1$.
- $\# H^1_f(\mathbb{Q}, \rho_f(-1) \otimes E/\mathcal{O}) = \# \mathbb{F}$.

Then $\underset{\underset{R}{\rotatebox{90}{$\simeq$}}}{R_{\bar{\sigma}}^{\text{min, tcd}}} \simeq \mathcal{O}$ for $\bar{\sigma}$ $\overset{\text{non}}{\vee}$ ss. reducible constructed from $\sigma_{A,p}$ as in Klosin's talk.

<u>Proof strategy:</u> - adapt lattice construction s.t. $\bar{\sigma}|_{I_N} = \mathbb{1} \oplus \bar{\rho}_f \oplus \chi$.

- study N-minimal deformations
- red. ideal $I \subset R$ is principal

$- \# R/I \leq \# H^1_f(\mathbb{Q}, \rho_f(-1) \otimes E/\mathcal{O}) = \# \mathbb{F}.$

$\Rightarrow I$ is maximal

Together these give $R \xrightarrow{\sim} \mathcal{O}$

<u>Cor:</u> Under the assumptions of the theorem. If $\exists\ F \in S_2^{(2)}(K(N))$ a non-lift eigenform s.t.

(a) $\lambda_\ell(F) \equiv 1 + \ell + a_\ell(f) \pmod{p}$ for all $\ell \nmid pN$.

(b) $\sigma_{F,p}$ crystalline, N-min irred def

Then A is paramodular of level N.

<u>Example:</u> $N = 277 \quad p = 3$

$A = \mathrm{Jac}(C) \quad C: y^2 + y = x^5 - 2x^3 + 2x^2 - \cancel{x}.$

(a) Poor-Yuen prove $S_2^{(2)}(K(277))^{\text{non-lift}} = \mathbb{C} \cdot F.$

and $F \equiv SK(f) \pmod{3}$ for $f \in S_2^{-}(\Gamma_0(277))$

$\updownarrow$

E 277a

(b) Jorza $\Rightarrow \sigma_{F,p}|_{D_p}$ is crystalline