Siegel Zeros:

Let $D > 0$ and consider $\mathbb{Q}(\sqrt{-D}) = K$. Let h_K be the class number of K.

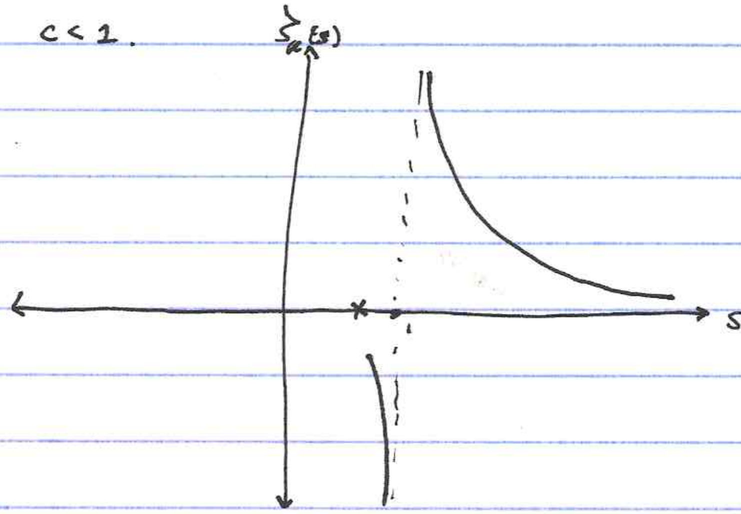
Gauss conjectured that $h_K = 1$ for exactly 9 K. This was proved by Stark in the 60's.

$$\zeta_K(s) = \zeta(s)\, L(s, \chi)$$

$$\operatorname*{res}_{s=1} \zeta_K(s) = L(1, \chi) = \frac{\pi h_K}{\sqrt{D}} > ?$$

Up to 1965 people were able to show $\frac{\pi h_K}{\sqrt{D}} > \frac{c\pi}{\sqrt{D}}$

for some $c < 1$.



So $\zeta_K(s)$ is negative for $s < 1$ unless there is a β so that $\zeta_K(\beta) = 0$. Such a β, if it exists, is called a Siegel zero.

<u>Fact</u>: $L(1, \chi) \gg (1 - \beta)$ (where $\gg$ means bigger than it times a constant). So how big is $(1-\beta)$?

Known: $1-\beta \gg \frac{1}{\sqrt{D}}$. $\Rightarrow \frac{h\pi}{\sqrt{D}} \gg \frac{1}{\sqrt{D}} \Rightarrow h \gg 1$.

Conj: $1-\beta \gg \frac{1}{\log D}$.

If this is true, then

$$\frac{\pi h}{\sqrt{D}} \gg \frac{1}{\log D} \Rightarrow h \gg \frac{\sqrt{D}}{\log D}.$$

Goals: ① $L(1,\chi) \gg 1-\beta$

② Siegel zeros repel.

③ Siegel zeros are practically unique amongst all automorphic L-series.

④ The only possible β must come from some $\mathbb{Q}(\sqrt{D})$

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^2 \, ds}{s(s+1)(s+2)} = \begin{cases} \frac{1}{2}\left(1-\frac{1}{x}\right)^2 & \text{if } x>1 \\ 0 & \text{if } 0<x\le 1 \end{cases}$$

$(x>1)$

$$\zeta_K(s) = \sum_{n\geq 1} \frac{a(n)}{n^s} \qquad a(1)=1,\ a(n)\geq 0.$$

Converges for $\mathrm{Re}(s)>1$.

$$\frac{1}{2\pi i}\int_{(2)} \frac{\zeta_K(s+\beta)\, x^s \, ds}{s(s+1)(s+2)} = \frac{1}{2\pi i}\int_{(2)} \sum \frac{a(n)\, x^s \, ds}{n^{s+\beta}\, s(s+1)(s+2)}$$

$\left(\frac{3}{4}<\beta<1\right)$

$$= \sum \frac{a(n)}{n^{\beta}} \frac{1}{2\pi i} \int_{(2)} \frac{\left(\frac{X}{n}\right)^s ds}{s(s+1)(s+2)} .$$

$$= \frac{1}{2} \sum_{n<X} \frac{a(n)}{n^{\beta}} \left(1-\frac{n}{X}\right)^2 . \gg 1$$

Now move the line of integration to $Re(s) = \frac{1}{2} - \beta \approx -1/2$.

Poles at $s=0$ and $s = 1-\beta$. (others are outside our box of integration.

$$\operatorname*{Res}_{s=0} = \frac{\zeta_K(\beta)}{2} .$$

$$\operatorname*{Res}_{s=1-\beta} = \frac{L(1,\chi)\, X^{1-\beta}}{(1-\beta)(2-\beta)(3-\beta)}$$

Thus,

$$1 \ll \frac{1}{2\pi i} \int_{(2)} \frac{\zeta_K(s+\beta)\, X^s\, ds}{s(s+1)(s+2)} = \frac{\zeta_K(\beta)}{2} + \frac{L(1,\chi)\, X^{1-\beta}}{(1-\beta)(2-\beta)(3-\beta)}$$

$$+ \frac{1}{2\pi i} \int_{(1/2-\beta)} \frac{\zeta_K(s+\beta)\, X^s\, ds}{s(s+1)(s+2)}$$

$$= O(X^{1/2-\beta}) \approx O(X^{-1/2})$$

$1-\beta \gg \frac{1}{\log D}$ and $\zeta_K(\beta) \leq 0$.

Thus, we can shift $\frac{\zeta_K(\beta)}{2}$ to the other side of the inequality to get

$$1 \ll \frac{L(1,\chi)\, X^{1-\beta}}{(1-\beta)(2-\beta)(3-\beta)}$$

$\Leftrightarrow$

$$\frac{1-\beta}{X^{1-\beta}} \ll L(1,\chi).$$

Let $x = D^4$. Then $X^{1-\beta} = D^{\frac{4c}{\log D}} \ll 1$.

If there is no zero, then

$$L(1,\chi) \gg \frac{1}{\log D}.$$

Now we want to show Siegel zeros repel.

Let $L(s)$ be any L-fctn and $G(s)$ any product of Gamma functions. (To be specific, $L(s) = \zeta_K(s)$ for K a number field.) $D =$ disc K

$$\left(\frac{D}{2^{?}\pi^{?}}\right)^{s/2} G(s)\,\zeta_K(s)\, s(1-s) = C \prod \left(1 - \frac{s}{\rho}\right)$$

where $\rho = \beta + i\gamma$, ρ zeros of $\zeta_K(s)$, $0 < \beta < 1$.

We have taking logarithmic derivatives

$$\frac{1}{2}\log D + \frac{G'(s)}{G(s)} + \overbrace{\frac{\zeta_K'(s)}{\zeta_K(s)}}^{<0} + \frac{1}{s} + \frac{1}{s-1}$$

$$= \sum \frac{2(s-\beta)}{(s-\beta)^2 + \gamma^2} > 0 \qquad \begin{matrix} s \text{ real} \\ s > 1. \end{matrix}$$

$$= \sum_{\rho = \beta = \text{real}} \frac{1}{1-\beta} + (>0)$$

$$\Longrightarrow \quad \frac{1}{2}\log D + \frac{1}{s-1} > \sum \frac{1}{1-\beta} \qquad (*)$$

Set $s = 1 + \frac{(1-\beta)}{M}$. So if there are two zeros, the RHS of $(*)$ is too big for the LHS.

$K = \#$ field $\quad K/\mathbb{Q}$ Galois with $G = \mathrm{Gal}(K/\mathbb{Q})$.

Then

$$\text{let } \zeta_K(s) = \prod_{\chi \in \hat{G}} L(s, \chi, K/\mathbb{Q})^{\deg \chi}.$$

If $\zeta_K(\beta) = 0$ and β is a simple zero, then all $L(s, \chi, K/\mathbb{Q})$ are analytic at $s = \beta$.

$L(s, f) = L$-series of f for f a Maass form

$$L(s, f) = \sum \frac{a(n)}{n^s} \qquad a(1) = 1.$$

We could also normalize so that $\langle f, f \rangle = 1$ instead of using $a(1) = 1$. Then

$$(*) \qquad 1 = \langle f, f \rangle = \iint_D |f(z)|^2 \frac{dx\,dy}{y^2} = \operatorname*{Res}_{s=1} \iint_D |f(z)|^2 E(z, s) \frac{dx\,dy}{y^2}$$

$$f(z) = \sum_{n \neq 0} \rho(n) \sqrt{y}\, K_{it}(\ldots)\, e^{2\pi i n x}$$

$$\rho(n) = a(n)\rho(1).$$

$$E(z, s) = \sum_{\Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^{-s} \qquad \operatorname*{Res}_{s=1} E(z, s) = \frac{3}{\pi}\,?$$

Then

$$(*) = \sum \frac{a(n)^2}{n^s}\, \Gamma\!\left(\frac{s}{2} + it\right) \Gamma(s/2 - it) \cdots$$

$$\sum \frac{a(n)^2}{n^s} = L(s, f \times f) = L(1, f, V^2)\, \zeta(s)$$

$$1 = \rho(1)^2 \, L(1, f, \nu^2) \, \overbrace{\Gamma(\tfrac{1}{2} + it)\Gamma(\tfrac{1}{2} - it)}^{e^{\pi|t|}}$$

$$\rho(1)^2 = \frac{e^{\pi|t|} \quad c}{L(1, f, \nu^2)}$$

Using the same methods as above one gets

$$L(1, f, \nu^2) > (1-\beta)$$
$$> \frac{1}{\log(\ \ )}$$

So the entire issue was controlled by a potential Siegel zero.
One uses automorphic forms on GL(3) to study this.

Rankin-Selberg L-functions at the special points:

Let $\{u_j\}_j$ be an orthonormal Hecke basis of Maass cusp forms in $L^2(SL_2(\mathbb{Z})\backslash\mathbb{H})$

$$\Delta u_j = \lambda_j u_j \qquad \lambda_j = s_j(1-s_j) \qquad s_j = \tfrac{1}{2} + it_j, \ t_j > 0$$

$$T_n u_j = \lambda_j(n) u_j$$

$$u_j(z) = (\cosh(\pi t_j))^{1/2} y^{1/2} \sum_{n\neq 0} v_j(n) K_{it_j}(2\pi|n|y)\, e(nx)$$

Next fix a normalized newform in $S_4(\Gamma_0(p))$, p prime.

$$Q(z) = \sum_{n=1}^{\infty} q(n) n^{3/2} e(nz), \quad q(1) = 1.$$

$$\Lambda(s, Q\otimes u_j) = \left(\frac{\sqrt{p}}{2\pi}\right)^{2s} \Gamma(2+s-it_j)\,\Gamma(2+s+it_j)\, L(s, Q\otimes u_j).$$

$$\Lambda(s, Q\otimes u_j) = \Lambda(1-s, Q\otimes u_j). \quad \text{(functional equation).}$$

Interested in the point $s_j = 1/2 + it_j$. and the value of $L(s, Q\otimes u_j)$ at the spectral point $s = s_j$.

This point is very similar to the role played by the central critical value $s = 1/2$ in the elliptic modular case.

Phillips-Sarnak: (1985)

$L(s_j, Q\otimes u_j) \neq 0 \Rightarrow$ the cusp form u_j will be dissolved by the deformation generated by Q on the Teichmuller space of $\Gamma_0(p)$.

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(1) $$\sum_{t_j \le T} |v_j(1)|^2 |L(s_j, Q\otimes u_j)|^2 = a_0 T^2 \ln T + O(T^2 \ln\ln T)$$

(Deshouillers - Iwaniec 1986)

$a_0 = 4\pi^{-2} \operatorname{Res}_{s=1} L(s, Q\otimes Q)$.

(2) $$\sum_{t_j \le T} |v_j(1)|^2 L(s_j, Q\otimes u_j) = 2\pi^{-2} T^2 + O(T^{15/8})$$ (L. 1993).

(1), (2) $\Rightarrow$

$$\left(\max_{t_j \le T} |v_j(1)|^2\right) \# \{ t_j \le T ; L(s_j, Q\otimes u_j) \ne 0 \} \gg \frac{T^2}{\sqrt{\ln T}} \qquad (*)$$

(1994) Hoffstein - Lockhart

$$|v_j(1)|^2 = \frac{1}{\langle u_j, u_j\rangle} = \frac{\pi}{2} \frac{1}{L(1, \operatorname{sym}^2(u_j))}.$$

$$|v_j(1)|^2 \ll_\varepsilon t_j^\varepsilon.$$

use this to get $\quad LHS\,(*) \gg_\varepsilon T^{2-\varepsilon}$.

(2001) L.

$$\# \{ t_j \le T : L(s_j, u_j\otimes Q) \ne 0 \} \gg T^2$$

- distribution of $L(1, \operatorname{sym}^2 u_j)$.
- Selberg's modified techniques

V. Blomer: Can you improve the error term in (1)?

More specifically, can we get LHS to be

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$$= a_0 T^2 \ln T + a_1 T^2 + O(T^{2-\delta}).$$

(2011 L.) Can get this with $\delta = 1/6$.