Funke
8-8-12
PS 1

Borcherds products and singular theta lifts for orthogonal and unitary groups:

I. Let $V/\mathbb{Q}$ be a quadratic space with signature $(p,2)$ with bilinear form $(\,,\,)$.

$G = SO_0(V(\mathbb{R})) \simeq SO_0(p,2)$ (connected comp. of identity)

$\cup$

K max. compact $\simeq SO(p)\times SO(2)$

$D = G/K$ symm. space.

$D \cong \{ Z \subset V(\mathbb{R}) : \dim Z = 2,\ (\,,\,)|_Z < 0 \}$. Hermitian structure $\dim_{\mathbb{C}} D = p$.

$\cong \{ \text{majorants } (\,,\,)_Z \}$

$V = Z \oplus Z^\perp$

$\overset{\psi}{x} = x_Z + x_{Z^\perp}$ $\quad (x,x)_Z = (x_{Z^\perp}, x_{Z^\perp}) - (x_Z, x_Z)$.

$L \subset V$ even lattice, $\Gamma \subseteq \mathrm{Stab}(L)$ in G

For today (for ease) we assume L is unimodular, i.e.

$L = L^\#$ = dual lattice.

$X = \Gamma \backslash D$ quasi projective variety

Special Cycles/Divisors

$x \in L$, $(x,x) > 0$

$D_x = \{ z \in D,\ x \perp z \} \cong D_{p-1,2}$ complex codim 1.

$\Gamma_x = \mathrm{Stab}(x) \cap \Gamma$

$$Z(x) = \Gamma_x \backslash D_x \longrightarrow \Gamma \backslash D = X$$

$$Z(N) = \sum_{\substack{x \in L \\ (x,x) = 2N > 0 \\ \bmod \Gamma}} Z(x).$$

Examples: 1) $(1,2)$: $G \simeq SL_2(\mathbb{R})$

$D \cong \mathbb{H}$. , $Z(N)$ are CM points of disc. $-N$.

2) $(2,2)$: $D \cong \mathbb{H} \times \mathbb{H}$

$D_x \cong \mathbb{H}$, $Z(N) = T_N$ Hirzebruch-Zagier divisors

3) $(3,2)$: $D \cong \mathbb{H}_2$ $D_x \cong \mathbb{H} \times \mathbb{H}$, $Z(N)$ Humbert surfaces.

II. dual pair $O(p,2) \times SL_2(\mathbb{R}) \subseteq Sp(p+2)$

V W $V \otimes W$

Weil representation ω : action of $O(p,2) \times SL_2(\mathbb{R})$

$(G' = SL_2(\mathbb{R}))$ acts on $S(V(\mathbb{R}))$ = Schwartz space.

$\varphi_0(x,z) = e^{-\pi(x,x)_z} \in [S(V(\mathbb{R})) \otimes C^\infty(D)]^G \cong S(V(\mathbb{R}))^K$

$K' = SO(2) \cong U(1)$ max. compact in $G' = SL_2(\mathbb{R})$.

$\omega(k')\varphi_0 = \det(k')^{\frac{p-2}{2}}\varphi_0$

Form a theta function

$$\Theta(\tau, z, \varphi_0) = \sum_{x \in L} \varphi_0(x, z, \tau)$$

$$\varphi_0(x,z,\tau) = v\,\varphi_0(\sqrt{v}\,x)\,e^{\pi i (x,x)\mu}.$$

$\tau \in \mathbb{H}$, $\tau = \mu + iv$

$\in \text{NonHol-}M_{\frac{p-2}{2}} \otimes C^\infty(X)$

$\text{NonHol-}M_{\frac{p-2}{2}}$ = non-holomorphic modular forms in τ; $C^\infty(X)$: in z

We now consider the singular theta lift / Borcherds lift.

Let $f \in H_{1-p/2}(SL_2(\mathbb{Z}))$ weak Harmonic Maass forms
($\xi(f) \in S_{p/2+1}$)

$$\phi_B(f, z) := \int_{SL_2(\mathbb{Z})\backslash \mathbb{H}}^{reg} \Theta(z, \tau, \varphi_0)\, f(\tau)\, d\mu(\tau)$$

- needs to be regularized
- logarithmic singularities along special divisors
- roughly : For $f \in M^!_{1-p/2}$,

$$\log \| \Psi_B(f, z)\| = \phi_B(f, z).$$

$F_N(\tau) = F_N(\tau, s = s_0)$

Hejhal (Maass) ... Poincare Series

of weight $1 - p/2$ s.t. $\xi_{1-p/2}(F_N) = P_N$ = N^{th} Poincare series.

Then

$\phi_B(F_N, z)$ is a Green's function for $Z(N)$, i.e.,

$$\int_{\Gamma\backslash D} \phi_B(F_N, z)\, dd^c \omega + \int_{Z(N)} \omega \qquad \omega \in \Omega_c^{2p-2}(X).$$

$$= \int_{\Gamma\backslash D} \Lambda_B(F_N, z) \wedge \omega$$

where $\Lambda_B(F_N, z) = \underbrace{\frac{-1}{2\pi i}\, \partial\bar\partial}_{dd^c}\, \phi_B(z, \tau)$.

III. KM-lift

$$\varphi_{KM} \in [S(V(\mathbb{R})) \otimes \Omega^{1,1}(D)]^G$$

$$\cong [S(V(\mathbb{R})) \otimes \Lambda^{1,1}(?)]^K$$

$\Theta(z,\tau,\varphi_{KM}) \in NHol\, M_{\frac{p}{2}+1} \otimes \Omega^{1,1}(X)$ s.t.

N^{th} Fourier coeff. of $\Theta(\varphi_{KM})$ is a Poincare dual form for $Z(N)$, also

$$\Lambda_{KM}(P_N, z) = (\Theta(\varphi_{KM}), P_N(\tau))_{Pet.}$$

is a Poincare dual form on $Z(N)$ ($PD(Z(N))$).

<u>Theorem (Brunier-F.):</u> $\Lambda_B(f,z) = \Lambda_{KM}(z, \xi(f)) + a_0 \Omega$

Kahler form

The key feature in the proof of this result is that

$$L_\tau \varphi_{KM}(x,z,\tau) = -dd^c \varphi_0(x,z,\tau).$$

lowering operator

on z

IV. Kudla's Green's function:

$$\xi(x,z) = -\int_1^\infty \varphi_0(\sqrt{t}\,x, z)\, e^{\pi(x,x)t} \frac{dt}{t}$$

Then the geometric properties of φ_B reduce to those of $\xi(x,z)$.

$$\begin{array}{ccc} \xi(x,z,\tau) & \xrightarrow{dd^c} & \varphi_{KM}(x,z,\tau) \\ L \downarrow & \circlearrowleft & \downarrow L \\ -\varphi_0(x,z,\tau) & \xrightarrow{dd^c} & \text{same thing.} \end{array}$$

One can now look at the same ideas only using (p,q) for the signature.