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p-adic L-functions and the eigencurve:

$N \geq 1$, $p \nmid N$ prime, $k_0 \geq 0$

$\Gamma = \Gamma_1(N)$, $\Gamma_0 \subseteq \Gamma_0(pN) \subseteq \Gamma$ $\quad \Gamma_1 = \Gamma_0(p) \cap \Gamma$

Motivating Thm: (Hida, Coleman) Let $f \in M_{k_0+2}(\Gamma_0)$ a Hecke eigenform, $f | U_p = \alpha_p f$, $0 \leq \mathrm{ord}_p(\alpha_p) < k_0 + 1$.

Then $\exists\, B \subseteq \mathcal{W} = \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$, $k_0 \in B$, B an affinoid disk and $\mathbb{f} \in A(B)[[q]]$,

$\mathbb{f} = \sum_{n \geq 0} \alpha_n q^n$ s.t.

1) $\mathbb{f}_k := \sum \alpha_n(k) q^n \quad k \in B \cap \mathbb{Z}_{\geq 0}$
then $\mathbb{f}_k$ is in $M_{k+2}(\Gamma_0)$

2) $\mathbb{f}_{k_0} = f$

Examples: 1) $\mathbb{E}_k := \frac{1}{2} \zeta_p(-1-k) + \sum_{n \geq 1} \sigma^*_{k+1}(n) q^n$

$$\sigma^*_{k+1}(n) = \sum_{\substack{d | n \\ p \nmid d}} d^{k+1}.$$

2) $f = q \prod_{n \geq 1} (1-q^n)^2 (1-q^{11n})^2 \in S_2(\Gamma_0(11))$.

$$\Delta = q \prod_{n \geq 1} (1-q^n)^{24} \in S_{12}(\Gamma(1)).$$

These are congruent mod 11.

$\Delta_\alpha = \Delta(z) - \beta \Delta(pz)$ is congruent mod 11 to Δ and fits in family.

α = unit root of char. poly of U_p on Δ.

The eigencurve:

This puts all this together.

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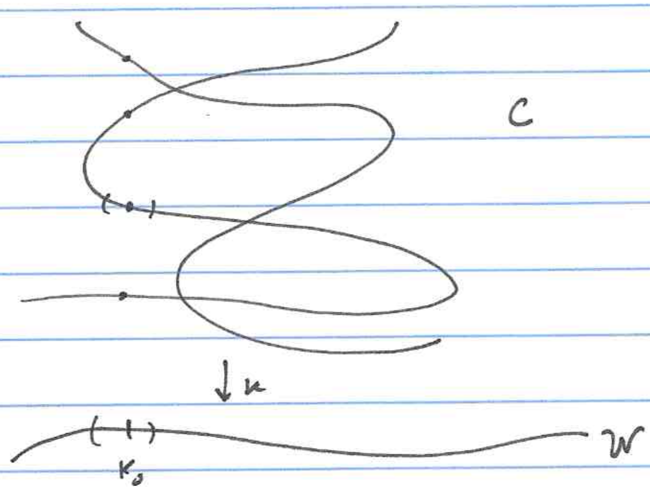
<u>Thm (Coleman-Mazur-Buzzard):</u> $\exists$ C a p-adic rigid analytic curve,

$$C \xrightarrow{\kappa} \mathcal{W}$$

κ locally finite rigid analytic map



s.t.

(a) $\exists$ 1-1 correspondence (for any fixed $\kappa_0 \in \mathcal{W}$) $K/\mathbb{Q}_p$ finite ext.

$$\left\{ \begin{array}{c} x \in C(K) \\ \text{s.t. } \kappa(x) = \kappa_0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{overconvergent eigenforms} \\ f_x \in M^{\dagger}_{\kappa_0+2}(\Gamma_0, K) \\ \alpha_p(f_x) \neq 0 \end{array} \right\}$$

(b) $\exists$ rigid analytic fctns

$$\alpha_n : C \longrightarrow \mathbb{C}_p \quad \text{s.t.}$$

$\forall\, x \in C(K)$,

$$f_x = \sum_{n \geq 0} \alpha_n(x) q^n$$

Moreover, C is smooth and unramified $/\mathcal{W}$ at every classical point x of ~~exceptional slope~~ noncritical slope.

(smooth but ramified at critical$^{\varepsilon}$ slope by recent work of Bellaïche)

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Thm (Amice-Velu): If $0 \le \text{slope}(f) < k_0+1$ then $\exists !$
$L_p(f,\cdot)$ satisfying
1) interpolation
2) $L_p(f)$ * is tempered of order slope(f).

$$L_p(f,\cdot) : \mathcal{W} \longrightarrow \mathbb{C}_p$$

dots = integral pts

interpolation is at dots.

tempered is a statement about the growth rate of L_p as discs increase

Interpolation does not work at critical slopes.

Modular Symbols:

$\varphi \in \text{Symb}_{\Gamma_0}(M)$ $\qquad$ M a Γ_0-module

$$\| $$

$\text{Hom}_{\Gamma_0}(\text{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)$

Thm (Ash, S.): If $\#\text{Tor}(\Gamma)$ acts invertibly on M, then there is a canonical isom.

$$H^1_c(\Gamma_0, M) \cong \text{Symb}_{\Gamma_0}(M)$$

$\varphi \in H^1_c(\Gamma_0, M)$

$$\varphi(r \to s) := \varphi(\{s\} - \{r\}) \in M$$

$\Lambda(\varphi) = \varphi(\infty - 0)$ Universal L-value

$f \in S_{k_0+2}(\Gamma_0)$

$\omega_f = 2\pi i f(z)(zX+Y)^{k_0}\,dz$

$\in \Omega^1(\mathfrak{h}) \otimes \underbrace{\mathrm{Sym}^{k_0}(\mathbb{C})}_{L_{k_0}}$

$\Psi_f(r \to s) = \int_r^s \omega_f \in L_k$

$\Lambda(\Psi_f) \in L_k$.

Locally analytic distributions:

$\mathcal{A}(\mathbb{Z}_p)$ = locally analytic fctns on $\mathbb{Z}_p$. (Amice)

$\mathcal{D}(\mathbb{Z}_p) = \mathcal{A}(\mathbb{Z}_p)^*$ = cont. linear functionals on $\mathcal{A}(\mathbb{Z}_p)$.

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Z}_p) & \ni & \mu \\ \downarrow \text{rest.} & & \updownarrow \\ \mathcal{D}(\mathbb{Z}_p^\times) & & \mu|_{\mathbb{Z}_p^\times} \end{array}$$

Def: $L_p(\mu, s) = \int_{\mathbb{Z}_p^\times} t^{s-1}\, d\mu(t) \in \mathbb{C}_p$

rigid analytic for $s \in \mathcal{W}$

Thm (Amice): $\mathcal{D}(\mathbb{Z}_p^\times) \xrightarrow{\sim} A(\mathcal{W})$

$\mu \longmapsto L_p(\mu, \cdot)$

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Γ_0 acts on $\mathcal{A}(\mathbb{Z}_p)$, depending on $\kappa \in \mathcal{W}$.

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0$$

$$(\gamma f)(z) = (a+cz)^{\kappa} f(z\gamma)$$

The action $\Gamma_0 \circlearrowright \mathcal{A}_\kappa(\mathbb{Z}_p)$ induces an action via duality on $\mathcal{D}_\kappa(\mathbb{Z}_p)$.

Study: $H^1_c(\Gamma_0, \mathcal{D}_\kappa(\mathbb{Z}_p))$.

$$\underset{\substack{\uparrow \\ \text{polys of} \\ \text{deg } k_0}}{\mathcal{P}_{k_0}} \subseteq \mathcal{A}_{k_0}$$

$$L_{k_0} = \mathcal{P}_{k_0}^* \longleftarrow \mathcal{D}_{k_0}$$

$$k_0 \geq 0 \Rightarrow H^1_c(\Gamma_0, \mathcal{D}_{k_0}) \longrightarrow H^1_c(\Gamma_0, L_{k_0})$$

If $0 \leq h < k_0 + 1$ then
$$H^1_c(\mathcal{D}_{k_0})^{(<h)} \cong H^1_c(L_{k_0})^{(<h)}$$

$$f \in S_{k_0+2}(\quad)^{(<k_0+1)}$$

$$\begin{array}{ccl} \varphi_f^{\pm} & \in & H^1_c(\Gamma_0, L_{k_0})^{(<k_0+1)} \\ \updownarrow & & \quad \wr\Vert \\ \Phi_f^{\pm} & \in & H^1_c(\Gamma_0, \mathcal{D}_{k_0})^{(<k_0+1)} \end{array}$$

$$\mu_f := \Lambda(\Phi_f) \in \mathcal{D}_{k_0}$$

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$$L_p(f,1) := L_p(\mu_f, s)$$

This recovers the p-adic L-function given above, but this has the possibility of generalization.

<u>Def:</u> <u>Cohomological eigencurve:</u>

$$\mathcal{C}(K) := \{ \xi : \mathcal{H} \longrightarrow K : \xi \text{ occurs in } H_c^1(\mathcal{D}_k) \text{ some } k \ \& \ \xi(U_p) \neq 0 \}$$

$\mathcal{H}$ = Hecke Algebra

<u>"Spreading out"</u>

$\mathcal{A} = \mathcal{A}(\mathbb{Z}_p^\times \times \mathbb{Z}_p) \circlearrowleft \Gamma_0$

$\mathcal{D} = \mathcal{A}^* =$ cts linear functionals on $\mathcal{A}$.

$\mathcal{D}(\mathbb{Z}_p^\times) \circlearrowright \mathcal{D} \circlearrowleft \Gamma_0$

$\mathcal{D}(\mathbb{Z}_p^\times) = A(\mathcal{W})$ = rigid analytic fctns on $\mathcal{W}$

Try to make a $\Phi \in H_c^1(\mathcal{D})$ that is an eigensymbol.

If one could do this, one would set

$\mu = \Lambda(\Phi)$ and

$$L_p(\Phi, k, s) := \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x^k \, (y/x)^{s-1} \, d\mu(x,y)$$

We can only make such a Φ in special cases.

$$\mathcal{D} \to \mathcal{D}_\Omega \longrightarrow \mathcal{D}_{k_0} \longrightarrow L_{k_0}$$

$\Omega \subseteq \mathcal{W}$ an affinoid nbd of k_0

$$\mathcal{D}_\Omega := \mathcal{D} \otimes_{\mathcal{D}(\mathbb{Z}_p)} A(\Omega)$$

Using Coleman's theory of orthonormalizable Banach modules and theory of Fredholm-Riesz on $H^1_c(\Gamma_0, \mathcal{D}_\Omega)$

$$P = \det(1 - TU_p, H^1_c(\mathcal{D}_\Omega)) \in \Lambda[\![T]\!]$$

Get P_Ω by restriction, and this lives in $A(\Omega)[\![T]\!]$.