Periods of Modular Forms:

We will be interested in period relations

V variety / F = a number field.
ω algebraic differential form on V rational over F.
$F \overset{\sigma}{\hookrightarrow} \mathbb{C}$
γ a topological cycle on $V^{\sigma}(\mathbb{C})$, period $= \int_\gamma \omega$.

Motivating problem: Tate conjecture:

$V_1, V_2 / F$

$Gal(\bar{F}/F)$ acts naturally on $H^k_{et}(V_1)$, $H^k_{et}(V_2)$.

Suppose there is a common irreducible Galois rep. in $H^k_{et}(V_1)$ and $H^k_{et}(V_2)$

The Tate conjecture $\Rightarrow$ a correspondence on $V_1 \times V_2$ that realizes this isom.

$$Z \subseteq V_1 \times V_2$$

if one knew this, then
$\Rightarrow$ relations between periods on V_1 ~~are~~ and periods on V_2.

- Can we prove such period relations without knowing the Tate conjecture?

Why would we want to do this?

1) Periods occur as the transcendental parts of special values of L-functions

2) Hope: gives some ideas on constructing algebraic cycles.

Examples: Langland's functoriality

Jacquet-Langlands Correspondence:

f classical modular form, $f \in S_2(\Gamma_0(N))$, a newform, $f = \sum a_n q^n$, $a_1 = 1$.

$K_f = \mathbb{Q}(a_n)$ is a number field.

$f \rightsquigarrow$ Galois representation:

λ any prime of K_f, $\rho_{f,\lambda}: Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K_{f,\lambda})$.

(Shimura) characterized by: $(\lambda | \ell)$.

1) $p \nmid N\ell$ $\rho_{f,\lambda}$ is unramified at p.

2) char. poly. of $\rho_{f,\lambda}(Frob_p) = T^2 - a_p T + p$.

$X_0(N)$ $J_0(N) = Jac(X_0(N))$ are used to construct $H^1_{et}(X_0(N), \mathbb{Q}_\ell)$ these Galois representations.

It can happen that f ~~transfers~~ transfers to an indefinite quaternion algebra/$\mathbb{Q}$. In that case, $\rho_{f,\lambda}$ can be realized on certain Shimura curves.

B quaternion algebra/$\mathbb{Q}$ (central simple algebra/$\mathbb{Q}$ that is 4-dim.) $x^2 = a$, $y^2 = b$, $xy = -yx$.

$$B \otimes \mathbb{Q}_v = \begin{cases} M_2(\mathbb{Q}_v) : \text{for all but finitely many } v \text{ (split)} \\ \text{Unique (up to isom) quaternion div. alg.} : \text{a finite set of } v, \text{ of even cardin. (ramified)} \end{cases}$$

Suppose B is indefinite: $B \otimes \mathbb{R} = M_2(\mathbb{R})$.
Let $\mathcal{O}$ be an order in B (subring of B that is rank $4/\mathbb{Z}$).

(e.g. 1) $B = M_2(\mathbb{Q})$, $\mathcal{O} = M_2(\mathbb{Z})$
2) $B = M_2(\mathbb{Q})$, $\mathcal{O}_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}$.)

Each quaternion alg. has a reduced norm and trace.
$x \mapsto x^\iota$ = main involution. Then $nr(x) = xx^\iota$
$tr(x) = x + x^\iota$.

Let $\Gamma = \{x \in \mathcal{O} : nr(x) = 1\}$ ($nr = \det$ for $M_2(\mathbb{R})$)

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[rr, bend right] & B \otimes \mathbb{R} \arrow[r, "\sim"] & M_2(\mathbb{R}) \\ & & SL_2(\mathbb{R}) \end{tikzcd}$$

Γ is discrete in $SL_2(\mathbb{R})$.
$X_B = \Gamma \backslash \mathfrak{H}$ (e.g. $\mathcal{O}_N$; $\Gamma = \Gamma_0(N)$ and the curve is $Y_0(N)$).

If $B \neq M_2(\mathbb{Q})$, then $\Gamma \backslash \mathfrak{H} = X_B$ is already compact. Can define modular forms, Hecke operators in usual way. However, since there are no cusps, q-expansions are not available.

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$X_B = \Gamma \backslash \mathfrak{H}$ is a complex curve.

Shimura: X_B has a canonical model $/\mathbb{Q}$.

characterized as follows

$$\underset{\text{imag quad}}{K} \hookrightarrow B \qquad B \otimes \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R})$$

$$K^\times \hookrightarrow GL_2^+(\mathbb{R})$$

$K^\times$ acts on $\mathfrak{H}$. This action has a unique fixed point τ.

$$\mathfrak{H} \longrightarrow \Gamma \backslash \mathfrak{H}$$
$$\tau \longmapsto [\tau] =: P_\tau$$

We require P_τ be algebraic and that $Gal(\overline{\mathbb{Q}}/K)$ acts on P_τ in a prescribed way.

X_B, modular form of wt 2, g an eigenform for Hecke algebra.

Do we get new systems of Hecke eigenvalues?

Eichler, J-L.: No, all systems of eigenvalues appear on $M_2(\mathbb{Q})$. (Recall $M_2(\mathbb{Q})$ gave classical forms and $Y_0(N)$)

J.-L. criterion for a system of Hecke e.v.'s on $M_2(\mathbb{Q})$ to appear on B.

The same construction as before will give

$\rho_{g,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K_{g,\lambda})$.

But $g \longleftrightarrow f$ on $X_{M_2(\mathbb{Q})}$. One gets $\rho_{f,\lambda} \simeq \rho_{g,\lambda}$.

g lives on $H^1(X_B)$ and f on $H^1(X_{M_2(\mathbb{Q})})$.

So the Tate conj. $\Rightarrow$ cycle on $X_B \times X_{M_2(\mathbb{Q})}$ that realizes this.

Faltings $\Rightarrow$ this is true.

1) There is no known canonical construction
2) What if wt $f > 2$. (Scholl: Motives)
3) Replace $\mathbb{Q}$ by a totally real field F. (Hilbert modular forms)

F = totally real field (e.g. $F = \mathbb{Q}(\sqrt{D})$, $D>0$, $F = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$.)

Hilbert modular forms: $M_2(F)$

$[F:\mathbb{Q}] = d$ $\quad \Sigma_{F,\infty}$ = set of inf. places of F. $= \{v_1, \ldots, v_d\}$

$(k_1, \ldots, k_d)$ weights.

Restrict to the case $(2, 2, \ldots, 2)$.

B quat. alg. $/F$ $\quad X_B$: Shimura variety assoc. to B.

$$B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \underset{\tau_1, \ldots, \tau_n}{M_2(\mathbb{R})^n} \times \mathbb{H}^{d-n}$$

X_B has dimension: n.

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X_B is defined over a reflex field F_B

$$\begin{tikzcd}[column sep=small] & \overline{\mathbb{Q}} \arrow[r, "v_1"] \arrow[d, dash] \arrow[dr, dash] & \mathbb{C} \\ & F \arrow[d, dash] \arrow[r, dash] & F^{gal} \arrow[d, dash] \\ & \mathbb{Q} \arrow[r, dash] & F_B \end{tikzcd}$$

$$Gal(\overline{\mathbb{Q}}/F_B) = \{ \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma\{\tau_1, \ldots, \tau_n\} = \{\tau_1, \ldots, \tau_n\} \}$$

$$Hom(F, \overline{\mathbb{Q}}) = \Sigma_{F,\infty}.$$
$$\sigma_i \longmapsto v_i$$

e.g. 1) if B is split at v_1 and ramified at $v_2, \ldots, v_d$, then $F_B = F$.

e.g. 2) if B is split at v_i and ramified at $v_j \neq v_i$, $F_B = \sigma_i F$.

Suppose f is a Hilbert modular newform

$$\rho_{f,\lambda} : Gal(\overline{\mathbb{Q}}/F) \longrightarrow GL_2(K_{f,\lambda})$$

If you had a quat. alg. B as in e.g. 1), then X_B/F dim 1, $H^1(X_B)$

General B: X_B/F_B natural rep. in middle dim. cohom. has dim 2^n. Can be constructed from $\rho_{f,\lambda}$ by a "tensor induction",

$$\underline{X_B} \qquad \underline{X_{\tau_1} \times \cdots \times X_{\tau_n}}$$
$$\rho_B|_{Gal(\overline{\mathbb{Q}}/F^{gal})} \xrightarrow{\sim} \rho_{\tau_1} \otimes \cdots \otimes \rho_{\tau_n}$$

ρ_{τ_i} constructed from a B_{τ_i} (split at τ_i, ram. elsewhere)

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1) $X_B \cong X_{\tau_1} \times \cdots \times X_{\tau_n}$

2) Suppose B, B' have complimentary ramification at ∞.
$(X_B \times X_{B'}) \times X_{M_2(F)}$.

3) Suppose B, B' have same ramification at ∞.
$X_B \times X_{B'}$.

These are examples where we expect the Tate conj. to hold.

Shimura's Conjecture:

How are $\langle f_B, f_B \rangle$ related as B varies?

For 3) above, $\langle f_B, f_B \rangle \sim_{\bar{\mathbb{Q}}^\times} \langle f_{B'}, f_{B'} \rangle$

2) $\langle f_B, f_B \rangle \cdot \langle f_{B'}, f_{B'} \rangle \sim_{\bar{\mathbb{Q}}^\times} \langle f, f \rangle$

equal up to $\bar{\mathbb{Q}}^\times$ multiple.

F = totally real field, $[F:\mathbb{Q}] = d$

$\Sigma_{\infty, F} = \{v_1, \dots, v_d\}$

B = quaternion algebra over F.

f Hilbert modular form of wt $(2, \dots, 2)$.

B_{v_i} = quaternion algebra split at v_i and ramified at v_j, $j \neq i$

(it may not exist, but we assume it does)

$X_{v_i} = X_{B_i}$ = Shimura curve.

Assume for now that f transfers to B_{v_i} $\forall i$.

B is split at $\{\tau_1, \dots, \tau_n\}$ ramified at other infinite places. $\{\tau_1, \dots, \tau_n\} \subseteq \{v_1, \dots, v_d\}$.

$X_B \times X_{\tau_1} \times \dots \times X_{\tau_n}$	Shimura's conj (see pg 10)
B, B' have complementary ramification at ∞ $(X_B \times X_{B'}) \times X_{M_2(F)}$	$\langle f_B, f_B \rangle \langle f_{B'}, f_{B'} \rangle \sim_{\overline{\mathbb{Q}}^\times} \langle f, f \rangle$ proved by Shimura.
B_1, B_2 two quat. algs. same ramification at ∞ $X_{B_1} \times X_{B_2}$	$\langle f_{B_1}, f_{B_1} \rangle \sim_{\overline{\mathbb{Q}}^\times} \langle f_{B_2}, f_{B_2} \rangle$

Consequences for period relations:

Petersson inner product:

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$F = \mathbb{Q}$, B indefinite, f on $\Gamma \hookrightarrow SL_2(\mathbb{R})$, wt 2.
How do we normalize f?
If $B = M_2(\mathbb{Q})$, use q-expansion.

$f \rightsquigarrow$ section of a line bundle
$f \longmapsto 2\pi i f(z) dz \quad \Gamma \backslash \mathfrak{H}$
$\in H^0(X_B, \Omega^1)$
X_B has a canonical model $/\mathbb{Q}$.
Can pick a multiple of f that is rational over K_f.
Can do better: Fix a prime p, $p \nmid$ level N, then X_B has an integral model over $\mathbb{Z}[\frac{1}{N}]$. So one can normalize f up to p-units in K_f.

The same ideas work over totally real fields.

$F = \mathbb{Q}$, Then the Petersson product is given by

$$\langle f, f \rangle = \frac{1}{\mathrm{vol}(\mathfrak{H}/\Gamma)} \int_{\Gamma \backslash \mathfrak{H}} f(z) \overline{f(z)} \frac{dx\,dy}{y^2}.$$

If you think adelically, i.e., f as a form on $B^\times(\mathbb{A})$ (or $GL_2(\mathbb{A})$),

$$\langle f, f \rangle = \int_{Z(\mathbb{A}) GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})} f(g) \overline{f(g)}\, d\mu$$

$\nwarrow$ Tamagawa measure.

These definitions generalize to Hilbert modular forms.

For $F = \mathbb{Q}$, we have

$$\langle f, f\rangle \sim_{\overline{\mathbb{Q}}^\times} \Omega_+ \Omega_- .$$

Consequences for Period relations:

Q: How are $\langle f_B, f_B\rangle$ related as B varies?

Conj. (Shimura): $\exists$ a set of invariants $c_{v_1}, \dots, c_{v_d} \in \mathbb{C}^\times$

s.t.

$$\langle f_B, f_B\rangle \sim_{\overline{\mathbb{Q}}^\times} \prod_{\substack{B \text{ split} \\ \text{at } v_i}} c_{v_i}$$

Michael Harris proved this conjecture under the following assumption:

(*) $\exists$ at least one finite place v at which π_v is discrete series.

J-L: $f \rightsquigarrow \pi$ auto rep. of $GL_2(\mathbb{A}_F)$

$$\pi = \otimes \pi_v$$

f transfers to to $B^\times \Leftrightarrow \Sigma_B \leq \Sigma(\pi)$

Σ_B = set of places where B is ramified.

$\Sigma(\pi) = \{ v : \pi_v \text{ is discrete series}\} \supseteq \Sigma_\infty$

$v | \infty$ automatic

$v \nmid p\infty \Rightarrow \pi_v$ is Steinberg or supercuspidal.

Harris uses theta correspondence for unitary groups.

$\Rightarrow$ can remove (*) with some work.

Joint work w/ Ichino: quaternionic unitary groups.

$\Rightarrow$ can prove Shimura's conj. w/o (*)

$\Rightarrow$ integrality issues.

<u>Q</u>: Can we say something more precise?

$F = \mathbb{Q}$, $f \in S_k(\Gamma_0(N))$ $\quad N = N^+ N^-$

disc $B = N^-$

Like to formulate a conjecture.

Need to study congruences of modular forms:

(Put up Cremona's Table)

258 D, E is an example of congruence between newforms of the same level.

258A: $+ \ + \ 1 \ -5 \ 1 \ -3 \ 0 \ -7 \ldots$ evs.

129A: $0 \ + \ -2 \ -2 \ -5 \ 3 \ -3 \ 2 \ldots$

These are congruent mod. 3.

Can we predict such congruences?

Hida, Ribet, Wiles, Taylor-Wiles:

- Hida: $\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}$ is a measure of congruences: N = level of f.

$\ell \mid \frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-} \Leftrightarrow \exists\, g$ of level $\mid N$ s.t. $f \equiv g \pmod{\ell}$.

ie, $\bar\rho_{f,\lambda} \simeq \bar\rho_{g,\lambda}$.

$\langle f,f\rangle \doteq L(\mathrm{Sym}^2 f, 2) = L(\mathrm{ad}^0 f, 1)$

$L_p(s,f) = (1-\alpha_p p^{-s})(1-\beta_p p^{-s})$

$L_p(\mathrm{ad}^0 f, 1) = (1-{}^{\alpha_p}/_{\beta_p}\, p^{-s})(1-{}^{\beta_p}/_{\alpha_p}\, p^{-s})(1-p^{-s})$.

- Ribet: Level-raising / level-lowering:

f level N, $p \nmid N$

Can you find g of level Np s.t. $f \equiv g \pmod{\lambda}$? $\lambda \mid \ell$

Yes, iff $\ell \mid L_p(\mathrm{ad}^0 f, 1)$

f level pN; can you find g of level N s.t.

$f \equiv g \pmod{\ell}$

$\bar\rho_{f,\lambda} \simeq \bar\rho_{g,\lambda}$

If $\bar\rho_{f,\lambda}$ is unramified at p, then this is true

$f \longleftrightarrow$ isogeny class of elliptic curves, N sq. free, $p \mid N$.

$\bar\rho_{f,\lambda}$ irred is unramified at $p \Leftrightarrow \ell \mid c_p$ = order of the component group of the Neron model of E at p (E any curve in the isogeny class). This follows from the Tate parameterization.

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- Wiles, Taylor-Wiles:

Wiles: η-invariant (precise measure of congruences)

- η-inv $= \dfrac{\langle f, f\rangle}{\Omega_f^+ \Omega_f^-}$ (precise version of Hida)

- η-inv $\longleftrightarrow$ order of a Selmer group.

Used in proof of FLT.

$f \in S_2(\Gamma_0(N))$ newform. $\longleftrightarrow E$

B quat. alg. indef. $/\mathbb{Q}$ of disc $N^- \mid N$, $N = N^+ N^-$
X_B associated to an Eichler order of level N^+ in B.

Assume $\bar{\rho}_{f,\lambda}$ is irreducible (λ is not Eisenstein)
f_B, f normalized up to l-adic units.

<u>Prop:</u> $\dfrac{\langle f, f\rangle}{\langle f_B, f_B\rangle} \sim_\lambda \prod_{\substack{p \mid d_B \\ p \mid N^-}} c_p$ (up to λ-adic units)

True for abelian variety quotients.

$$\langle f_B, f_B\rangle \sim_\lambda \frac{\langle f, f\rangle}{\prod_{p \mid d_B} c_p} = \frac{L(1, \mathrm{ad}^0 f)}{\prod_{p \mid d_B} c_p}$$

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F tot. real.

$\exists$ $c_{v_1}, \ldots, c_{v_d}$ s.t.

$$\langle f_B, f_B \rangle \sim \prod_{\substack{B \text{ split} \\ \text{at } v_i}} c_{v_i} \sim \frac{\prod c_{v_i}}{\prod_{\substack{B \text{ ram} \\ \text{at } v_i}} c_{v_i}} \sim \frac{L(1, \mathrm{ad}^0 f)}{\prod_{\substack{B \text{ ram} \\ \text{at } v_i}} c_{v_i}}$$

Assume λ not Eisenstein.

<u>Conj:</u> $\exists$ a function $c : \Sigma(\pi) \longrightarrow \mathbb{C}^\times$ s.t.

$$v \longmapsto c_v$$

$$\langle f_B, f_B \rangle \sim_\lambda \frac{L(1, \mathrm{ad}^0 f)}{\prod_{v \in \Sigma_B} c_v}$$

- If v is inf., expect c_v are transcendental and alg. indep. except if f is a Base change
- If v is finite, expect c_v are (λ-adic) integers and count level-lowering congruences.

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Recall Conjecture:

F totally real, f Hilbert newform, π = auto. rep. assoc. to f.

$\exists$ invariants c_v, $v \in \Sigma(\pi)$ s.t.

$$\langle f_B, f_B \rangle = \frac{L(1, \mathrm{ad}^\circ \pi)}{\prod_{v \in \Sigma(B)} c_v} \quad \text{(up to Eis. primes)}$$

- If v is infinite, c_v = transcendental
- If v is finite, c_v = alg. integer.
  (if $p \mid c_v$, then $f \equiv g \pmod p$ $v \nmid \mathrm{level}(g)$).

Notes:

Thm: Suppose $F = \mathbb{Q}$, $f \longleftrightarrow$ isogeny class of elliptic curves, $f \in S_2(\Gamma_0(N))$, N sq. free.

What are the c_v's? $\Sigma(\pi) = \{\infty\} \cup \{q : q \mid N\}$.

$$c_\infty = \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega}_E$$

E any elliptic curve in isogeny class, ω_E = Neron differential

$q \mid N$ c_q = order of component grp. of Neron model of E at q.

If E' is another elliptic curve in same isogeny class, $E \to\to E'$.

B definite: (X_B finite set of points)

$$\langle f_B, f_B \rangle = \frac{L(1, \mathrm{ad}^\circ \pi)}{c_\infty \prod_{q \mid d_B} c_q}$$

There are relations between $\langle f_B, f_B \rangle$. as B varies.

(Student project 1: compute this for totally definite quat. alg. over totally real fields).

(Displayed some numerical data here)

We would like to generalize this to:

- higher weight
- tot. real fields.

Since the geometry is hard (or impossible) to generalize, we come at it from a different perspective.

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Theta correspondence:

$F =$ # field, $\mathbb{A}_F$; $\mathbb{W}$ symplectic space

$\langle\ ,\ \rangle : \mathbb{W} \times \mathbb{W} \to F$ that is nondeg. and alternating.

Fix ψ an additive char. of $F \backslash \mathbb{A}_F$.

Let $Sp(\mathbb{W})$ be the symplectic group of $\mathbb{W}$.

($GSp(\mathbb{W})$: similitude group)

$$1 \to \mathbb{C}^\times \to Mp_\psi(\mathbb{W})(\mathbb{A}) \to Sp(\mathbb{W})(\mathbb{A}) \to 1$$

metaplectic group.

not defining this here.

Weil representation: $\omega_\psi : Mp_\psi(\mathbb{W})(\mathbb{A}) \longrightarrow Aut(*)$.

<u>Dual reductive pair:</u> (Howe)

(G_1, G_2) reductive groups, $G_1 \times G_2 \leq Sp(\mathbb{W})$

so that G_1 and G_2 are centralizers of each other. in $Sp(\mathbb{W})$.

$$\begin{array}{ccc} \widetilde{G_1}(\mathbb{A}) \times \widetilde{G_2}(\mathbb{A}) & \leq & \boxed{Mp_{\psi}(\mathbb{W})(\mathbb{A})} \leftarrow \text{has } \omega_\psi. \\ \downarrow & & \downarrow \\ G_1(\mathbb{A}) \times G_2(\mathbb{A}) & \leq & Sp(\mathbb{W})(\mathbb{A}) \end{array}$$

(Can use this to transfer functions from $G_1(\mathbb{A})$ to $G_2(\mathbb{A})$ and in the other direction)

<u>Example:</u> 1) W = symplectic space V = orthogonal space

$\mathbb{W} = W \otimes V$ is a symplectic space

$(Sp(W), O(V))$ is a dual reductive pair in $Sp(\mathbb{W})$.

2) K/F quad. ext. V_1, V_2 are unitary spaces over K.

V_1 K-v.s. equipped w/ $\langle\, , \rangle : V_1 \times V_1 \to K$

$$\langle \alpha x, \beta y \rangle = \bar{\alpha} \langle x, y \rangle \beta$$

V_1 Hermitian and V_2 skew-Hermitian

$\langle x, y \rangle = \overline{\langle y, x \rangle}$ $\qquad$ $\langle x, y \rangle = -\overline{\langle y, x \rangle}$.

$\mathbb{W} = V_1 \otimes_K V_2$ thought of as an F-v.s.

$\langle\, , \rangle = tr_{K/F}(\langle\, , \rangle_1 \otimes \langle\, , \rangle_2)$ skew-symm.

$(U_K(V_1), U_K(V_2)) \leq Sp(\mathbb{W})$ is a dual reductive pair.

We can use Weil rep. to construct an integrating kernel.

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$q \in A$ ← space W_ψ acts on $\rightsquigarrow \Theta_q(g_1, g_2)$

$(G_1, G_2) \in Sp(W)$

f_1 on $G_1(\mathbb{A})$

$$\Theta_q(f_1)(g_2) = \int f_1(g_1)\,\Theta_q(g_1, g_2)\,dg_1$$

Can do the same to get fctns on $G_2(\mathbb{A})$ as well.

Ex: (Shimizu correspondence) B quat. alg $/F$

$V = B$, $\langle x, y\rangle = xy^\iota + yx^\iota$ ι = main involution

W = 2-dim symp. space.

$(Sp(W), O(V))$ $(GSp(W), GO(V))$

$$\begin{matrix} & \| \\ & (GL_2, (B^\times \times B^\times)/F^\times) \end{matrix}$$

$F^\times \backslash (B^\times \times B^\times) \xrightarrow{\sim} GO(V)$ $(\alpha, \beta) \longmapsto (x \mapsto \alpha x \beta^{-1})$

↑ embedded diagonally

Forms on $(B_1^\times \times B_2^\times)/F^\times$ look like pairs (π_1, π_2)

s.t. the central characters $\omega_{\pi_1}, \omega_{\pi_2}$ satisfy

$$\omega_{\pi_1}\omega_{\pi_2} = 1.$$

π on $GL_2(\mathbb{A})$;

$$\Theta(\pi) = \begin{cases} 0 & \text{if } \pi \text{ doesn't transfer to } B^\times \\ \pi_B \times \pi_B^\vee & \pi_B = JL(\pi) \end{cases}$$

In our case, central chars. are trivial so $\pi_B^\vee \simeq \pi_B$.

Pick $f \in \pi$, and explicitly compute

$$\Theta_\varphi(f) = \alpha \cdot (f_B \times f_B).$$ (can pick φ to make this happen)

So the arithmetic info. is in α.

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$GL_2 \longrightarrow (B^\times \times B^\times)/F^\times$

$GL_2 \longleftarrow (B^\times \times B^\times)/F^\times$ (Easier to study)

$\beta \cdot f \longleftarrow f_B \times f_B$ You can compute explicitly with f.c.'s on left.

show: $\beta = \langle f_B, f_B \rangle$

<u>Seesaw Dual Reductive Pair</u> (Kudla)

$(G_1, G_2), (H_1, H_2) \subseteq Sp(\mathbb{W})$ dual reductive pair

$$\begin{tikzcd} G_1 \arrow[d, dash] \arrow[dr, dash] & H_2 \arrow[d, dash] \\ H_1 \arrow[ur, dash] & G_2 \end{tikzcd}$$

$$\begin{tikzcd} \Theta_\varphi(f_2) & \Theta_\varphi(f_1) \\ f_1 \text{ on } H_1 \arrow[ur] & f_2 \text{ on } G_2 \arrow[ul] \end{tikzcd}$$

lifts exist b/c they are dual reductive pairs.

$$\int f_1 \, \Theta_\varphi(f_2)\big|_{H_1} = \int \Theta_\varphi(f_1)\big|_{G_2} f_2 \qquad \left(\begin{array}{c}\text{seesaw}\\ \text{duality}\end{array}\right)$$

<u>Ex:</u> V = orthogonal, W symplectic, $\mathbb{W} = V \otimes W$

$(O(V), Sp(W)) \subseteq Sp(\mathbb{W})$

$V = V_1 \oplus V_2$ (sum of orthogonal spaces)

$$\begin{tikzcd} Sp(W) \times Sp(W) \arrow[d, dash] \arrow[dr, dash] & O(V) \arrow[d, dash] \\ Sp(W) \arrow[ur, dash] & O(V_1) \times O(V_2) \end{tikzcd}$$

Applying this to our situation:

$$\alpha \langle f_B \times f_B, f_B \times f_B \rangle = \beta \langle f, f \rangle = \langle f_B, f_B \rangle \langle f, f \rangle$$

$$\Rightarrow \quad \alpha = \frac{\langle f, f \rangle}{\langle f_B, f_B \rangle}.$$

As we get the ratio we are interested in via this theta lift. Can we say anything about α now?

$$\begin{tikzcd}
 & & (B^\times \times B^\times)/F^\times = GO(V) \\
 & & \alpha(f_B \times f_B) \\
GL_2 \arrow[uurr, dash] & & \\
 & f \arrow[uur] &
\end{tikzcd}$$

$F = \mathbb{Q}$, B indefinite

form on $B^\times \rightsquigarrow$ sections of a line bundle on X_B

(usual modular forms: functions on pairs (E, ω))

X_B coarse moduli space, abelian surface with end. by B.

Check this on CM points: $K \hookrightarrow B$, $K^\times \hookrightarrow B^\times$
g

$$L_\chi(g) = \int_{\mathbb{A}_K^\times} g \cdot \chi \quad = \text{finite sums of values of } g \text{ twisted by } \chi^2$$

Pick a Hecke character χ of K of ∞ type $(2, 0)$ ← wt of g.

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<u>Criterion:</u> g is rational (integral) if $L_\chi(g)$ are rational (integral) up to periods of CM elliptic curves (CM periods)

$K \hookrightarrow B \qquad B = K \oplus K^\perp$
$\qquad\qquad\qquad V = V_1 \oplus V_2$

$$\begin{tikzcd}
GL_2 \times GL_2 \arrow[d, dash] \arrow[drr, dash] & & GO(V) \arrow[d, dash] \arrow[dll, dash] & = (B^\times \times B^\times)/F^\times \arrow[d, dash] \\
GL_2 = GSp(W) & & O(V_1) \times O(V_2) & (K^\times \times K^\times)/F^\times \arrow[l, mapsto, "(x,y)"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \alpha(f_B \times f_B) \arrow[d, dash] \\
f \arrow[urr] & & \chi \times \chi \arrow[ull]
\end{tikzcd}$$

$$\alpha \int (f_B \times f_B)(\chi \times \chi) = \int \Theta(\chi \times 1)\big|_{GL_2} f$$

$$= \int f \, \bar{\Theta}_\chi \, \Theta(1)$$

$$\alpha \frac{L_\chi(f_B)^2}{\Omega_{cm}} = \frac{\langle f \Theta(1), \Theta(\chi)\rangle}{\Omega_{cm}} = \text{value at } s = 1/2 \ \langle f B(1), \Theta(\chi)\rangle$$

value of Eis. series $E(s)$

'' Int. rep. for $L(s, f_B, \chi)$

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- Harris-Kudla L-~~series~~ value is rational
- P. (2005) L-values are integral (use main conj. in Iwas. th. for imag. quad. fields by Rubin)
- Factorization: p-adic families

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<u>A few remarks to start:</u>

- The Tate conjecture usually talks about $CH^i(X) \otimes \mathbb{Q}_\ell$. This can be approximated by $CH^i(X) \otimes \mathbb{Q}$. The integral Hodge/Tate conj. are not expected to be true.
    - Why do integral period relations exist?
    - Qn: Can we bound denominators in Tate conj.?

- Theta lifts
    - nonvanishing of theta lifts?
      Can be subtle : $\left\{ \begin{array}{l} \text{local conditions : } \varepsilon\text{-factors} \\ \text{global conditions: } L\text{-value} \end{array} \right.$
    - Algebraicity / integrality? $\longleftrightarrow$ Iwasawa theory.
    - Lift is a p-unit?

<u>Back to where we were:</u>

F = tot. real

two approaches $\left\{ \begin{array}{l} \text{Harris : unitary groups} \\ \text{Ichino/P : Quaternionic unitary grps.} \end{array} \right.$

<u>Harris:</u> B B' $\quad$ i = main involution

E/F CM field, $E \hookrightarrow B$, $E \hookrightarrow B'$ $\quad$ $\langle x,y \rangle = xy^i + \cancel{\ } yx^i$

$B = E + Ej$

Think of B as ~~an~~ a right E-v.s.

Can make B into a unitary space:

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$\langle\langle x,y\rangle\rangle = pr(x^{\iota}y^{\iota})$ $\qquad$ $pr: B = E + Ej \to E$.

$\langle\langle\ ,\ \rangle\rangle$ is an E-Hermitian form.

$$GU_E(B) \longleftrightarrow GO(B) \qquad \text{b/c } tr\langle\langle\ ,\ \rangle\rangle = \langle\ ,\ \rangle.$$

$$\overset{\shortparallel}{(B^\times \times E^\times)/F^\times} \qquad \overset{\shortparallel}{(B^\times \times B^\times)/F^\times}$$

A form on $GU_E(B)$ is a pair (π, χ), π form on $B^\times$, χ Hecke char. of $E^\times$ w/ $\omega_\pi \cdot \omega_\chi = 1$.

$$B, B' : \qquad U_E(B) \xrightarrow{\text{theta lift}} U_E(B') \qquad (\pi,\chi) \longmapsto \begin{cases} 0 \\ (\pi^\vee, \chi^{-1}). \end{cases}$$

Harris studies the arithmetic of this theta lift.

- ε-Factors
- $L(\tfrac{1}{2}, \pi, \chi)$ (Rankin-Selberg)
- Rallis inner product formula
  $$\langle \Theta_\varphi(f_B, \chi), \Theta_\varphi(f_B, \chi)\rangle = L(\tfrac{1}{2}, \pi, \chi)\langle f_B, f_B\rangle$$
- If B is at least as ramified as B' at ∞, then this lift is algebraic.

To see the algebraicity, one uses a seesaw arg again:

$$\begin{array}{ccc} U(B)\times U(B) & & U(B') \\ & \times & \\ U(B) & & U(V_1)\times U(V_2) \end{array} \qquad B' = E + Ej' = V_1 + V_2$$

To understand $U(V_1) \to U(B)$, one uses the Rallis inner product formula.

<u>Joint w/ Ichino:</u>

- Period integrals to L-values (Waldspurger/Tunnell-Saito)

F, π on $GL_2(\mathbb{A}_F)$, E/F, χ Hecke char. of E.

$\omega_\pi \cdot \omega_\chi = 1$. (central chars)

B quat. alg. $/F$ s.t. π transfers to π_B on $B^\times$

$f_B \in \pi_B$, suppose $E \hookrightarrow B$

$$\int f_B|_{\mathbb{A}_E^\times} \cdot \chi$$

Thm (Tunn/Saito/Wald): Given, π, χ, E, such that $\mathrm{sgn}\, L(s, f, \chi) = \varepsilon(\tfrac{1}{2}, \pi, \chi) = +1$, $\exists$ unique quat. alg. B s.t. π Transfers to π_B on $B^\times$ and $\exists\, f_B \in \pi_B$.

$$\left[\int f_B \chi\right]^2 \doteq L(\tfrac{1}{2}, \pi, \chi) \cdot \left(\begin{array}{c}\text{Normalization}\\ \text{factors}\end{array}\right)$$

$\left(\varepsilon_v(B) = \omega_{\pi,v}(-1) \cdot \varepsilon_v(\tfrac{1}{2}, \pi, \chi).\right)$ ~~[illegible]~~

- Triple product: π_1, π_2, π_3 on $GL_2(\mathbb{A}_F)$,, (Garrett, Harris-Kudla/ Watson/ Ichino-Ikeda,

$\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} = 1$.

Suppose $\varepsilon(\tfrac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) = +1$.

Then $\exists$ a unique quat. alg. B s.t. π_1, π_2, π_3 transfer to $\pi_1^B, \pi_2^B, \pi_3^B$, f_1^B, f_2^B, f_3^B

s.t.

$$\left[\int f_1^B f_2^B f_3^B\right]^2 \doteq L(\tfrac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)$$

Are these two formula related?

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$\pi_1 = \pi$, E/F CM, η_1, η_2 Hecke chars. of E,

$\pi_2 = \pi_{\eta_1}$, $\pi_3 = \pi_{\eta_2}$, $\omega_\pi \cdot \omega_{\eta_1} \cdot \omega_{\eta_2} = 1$. $\mathrm{Gal}(E/F) = \langle \rho \rangle$

$$L(s, \pi \otimes \pi_{\eta_1} \otimes \pi_{\eta_2}) = L(s, \pi, \chi_1)\, L(s, \pi, \chi_2) \qquad (*)$$

$B \qquad\qquad\qquad\qquad B_1 \qquad B_2$

where $\chi_1 = \eta_1 \eta_2$, $\chi_2 = \eta_1 \eta_2^\rho$. and each L-value has sign $+1$. Then $(*)$ gives:

$$[\cdot]^2 = [\cdot]^2 [\cdot]^2.$$

Can one remove the squares?

(The B is given by $\varepsilon_v(B) = \varepsilon_v(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)$ from triple product).

$B = B_1 \cdot B_2$ in Brauer group.

We know $E \hookrightarrow B_1, B_2$, and so $E \hookrightarrow B$.

$B_1 = E + \cancel{Ej_2}\ Ej_1$, $B_2 = E + Ej_2$

$\mathrm{Tr}(j_1) = \mathrm{Tr}(j_2) = 0$

$J_1 = j_1^2 \in F$, $J_2 = j_2^2 \in F$.

Think of B_1, B_2 as right E-spaces (unitary) Write the forms as $\langle , \rangle_1$, $\langle , \rangle_2$. . $B = E + Ej$, $j^2 = J_1 J_2$.

$$B_1 \otimes_E B_2 \circlearrowleft B$$

$\shortparallel$

4-dim as E-space w/ basis $1 \otimes 1$, $1 \otimes j_2$, $j_1 \otimes 1$, $j_1 \otimes j_2$.

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$(1\otimes 1)\cdot j := j_1 \otimes j_2$

$(1\otimes j_2)\cdot j := J_2(j_1\otimes 1)$ This gives an action

$(j_1\otimes 1)\cdot j := J_1(1\otimes j_2)$ of $\mathbb{B}$ on $B_1\otimes_E B_2$.

$(j_1\otimes j_2)\cdot j := J_1J_2(1\otimes 1)$

So $V = B_1\otimes_E B_2$ is a 2-dim right $\mathbb{B}$-v.s.

Pick $\alpha\in E$, $tr_{E/F}(\alpha)=0$, $\alpha\neq 0$.

$$\langle\,,\rangle = \alpha\langle\,,\rangle_1 \otimes \langle\,,\rangle_2 \quad \text{on } B_1\otimes_E B_2$$

This is a skew-Hermitian form.

One can find a B-skew Hermitian form $\langle\langle\,,\rangle\rangle$ on V s.t.

$pr.\langle\langle\,,\rangle\rangle = \langle\,,\rangle$.

$\langle\langle x\alpha, y\beta\rangle\rangle = \alpha^{\iota}\langle\langle x,y\rangle\rangle\beta \qquad \alpha,\beta\in B^\times$

$\langle\langle x,y\rangle\rangle = -\langle\langle y,x\rangle\rangle^{\iota}$ (skew-Hermitian)

$GU_B(V) = (B_1^\times \times B_2^\times)/F^\times$.

$W=B$, usual B-Hermitian form, $\langle x,y\rangle = xy^{\iota}$.

$GU_B(W) = B^\times$

$$GU_B(W) \longrightarrow GU_B(V)^\circ$$

$$B^\times \longrightarrow (B_1^\times\times B_2^\times)/F^\times$$

Thm: $\Theta(\pi_B) = \pi_{B_1}\times\pi_{B_2}$

$$f_B \longmapsto f_{B_1} \times f_{B_2}$$

$$\begin{array}{ccc} B^\times \times B^\times & & (B_1^\times \times B_2^\times)/F^\times \\ | & \times & | \\ B^\times & & (E^\times \times E^\times)/F^\times \end{array}$$

Seesaw duality $\Rightarrow$ equality of periods.

$$f_B \longmapsto \alpha(B_1, B_2)(f_{B_1} \times f_{B_2}).$$

Rallis inner product $\Rightarrow$

$$\alpha(B_1,B_2)^2 \langle f_{B_1}, f_{B_1}\rangle \langle f_{B_2}, f_{B_2}\rangle = \langle f_B, f_B\rangle L(1, \mathrm{ad}^\circ\pi)$$

$$\alpha(B_1,B_2)^2 \cdot \frac{L(1,\mathrm{ad}^\circ\pi)}{\prod_{v\in\Sigma_{B_1}} c_v} \frac{L(1,\mathrm{ad}^\circ\pi)}{\prod_{v\in\Sigma_{B_2}} c_v}$$

$$= \frac{L(1,\mathrm{ad}^\circ\pi)}{v\in\Sigma_B\ c_v} L(1,\mathrm{ad}^\circ\pi)$$

$$\Rightarrow \alpha(B_1,B_2)^2 = \frac{\prod_{v\in\Sigma_{B_1}} c_v \prod_{v\in\Sigma_{B_2}} c_v}{\prod_{v\in\Sigma_B} c_v}$$

$$= \prod_{v\in\Sigma_{B_1}\cap\Sigma_{B_2}} c_v^2$$

$$\Rightarrow \alpha(B_1,B_2) = \prod_{v\in\Sigma_{B_1}\cap\Sigma_{B_2}} c_v.$$

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<u>Conj B:</u> 1) If $\Sigma_{B_1,\infty} \cap \Sigma_{B_2,\infty} = \phi$, Then $\alpha(B_1, B_2) \in \overline{\mathbb{Q}}$ & it is a p-integer

2) If further, $\Sigma_{B_1} \cap \Sigma_{B_2} = \phi$, then $\alpha(B_1, B_2)$ is a p-unit.

<u>Conj B ⇒ Conj A:</u> We need to define c_v, $v \in \Sigma(\pi)$

Let $S \subseteq \Sigma(\pi)$, #S even, $c_S := \frac{L(1, ad^\circ \pi)}{\langle f_S, f_S \rangle}$.

Conj. B ⇒ $c_{S \uplus T} = c_S \cdot c_T$.

To define c_v: Pick r, s two other places in $\Sigma(\pi)$.

$$c_v^2 = \frac{c_{v,s}\, c_{v,t}}{c_{vs}}$$

$$\frac{c_{vs} c_{vt}}{c_{rs}} = \frac{c_{rs} c_{vs}}{c_{rs}} \qquad c_{vt}\, c_{rs} = c_{vs}\, c_{vt}.$$

So it remains to prove Conj. B.