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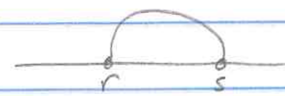
Overconvergent Modular Symbols:

Period integrals:

Let $f \in S_2(\Gamma_0(N))$. We want to consider the period integral:

$$2\pi i \int_r^s f(z)\,dz$$



$$(r,s \in \mathbb{P}^1(\mathbb{Q}))$$

$$2\pi i \int_{i\infty}^0 f(z)\,dz = L(f,1) \qquad L(f,s) = \sum_{n\geq 1} a_n n^{-s}$$

where

$$f(z) = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i z}$$

We can also handle twists this way:

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

$$L(f,\chi,s) = \sum_{n\geq 1} a_n \chi(n) n^{-s}$$

$$L(f,\chi,1) = c \cdot \sum_{a \,(\mathrm{mod}\, N)} \chi(a)\, 2\pi i \int_{i\infty}^{a/N} f(z)\,dz$$

Bogus argument (Heuristic):

$$2\pi i \int_{i\infty}^0 f(z)\,dz = 2\pi i \int_{i\infty}^0 \sum a_n q^n \,dz$$

$$= \sum \frac{a_n}{n} e^{2\pi i n z} \Big|_{i\infty}^0$$

$$= \sum \frac{a_n}{n} = L(f,1).$$

Numerical experiment:

$$f(z) \in S_2(\Gamma_0(11))$$

We can compute

$$2\pi i \int_r^s f(z)\,dz$$

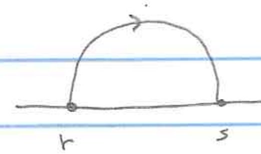
for a bunch of r's and s's.

(Included graphic plotting these. It is a lattice.)

Modular Symbols:

$$\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) = \text{degree 0 divisors on } \mathbb{P}^1(\mathbb{Q})$$

$$\{s\} - \{r\} \longleftrightarrow$$



$$\Psi_f : \Delta_0 \longrightarrow \mathbb{C} \qquad f \in S_2(\Gamma_0(N))$$

$$\{s\} - \{r\} \longmapsto 2\pi i \int_r^s f(z)\,dz \qquad \text{(extend linearly)}$$

$$\Psi_f \in \mathrm{Hom}(\Delta_0, \mathbb{C})$$

Easy fact:

$$\int_r^s f(z)\,dz = \int_{\gamma r}^{\gamma s} f(z)\,dz$$

$$\gamma \in \Gamma_0(N) \qquad \gamma z = \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\Rightarrow \quad \Psi_f(D) = \Psi_f(\gamma D)$$

$D \in \Delta_0$

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$$\Rightarrow \quad \Psi_f \in \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$$

Thus, we have a map

$$S_2(\Gamma_0(N)) \hookrightarrow \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$$
$$f \longmapsto \Psi_f.$$

There is an involution ι on $\mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$

$$(\iota(\varphi))(D) = \varphi\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} D\right)$$

So we can decompose $\varphi = \varphi^+ + \varphi^-$.

$\Gamma_0(11)$:

$f \rightsquigarrow \Psi_f^+$ takes values in $\mathbb{Z}\cdot\Omega^+$ $\qquad \Omega^\pm \in \mathbb{C}$.

$f \rightsquigarrow \Psi_f^-$ takes values in $\mathbb{Z}\cdot\Omega^-$

Anything else here?

Easier way to build symbols.. $\quad \Delta = \mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$

$$\mathrm{Hom}_{\Gamma_0(N)}(\Delta, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$$

φ = fctn on $\mathbb{P}^1(\mathbb{Q})$
constant on orbits of $\Gamma_0(N)$

$$\mathbb{P}^1(\mathbb{Q})/\Gamma_0(11) = \{\infty\} \cup \{0\}$$

where $\{\infty\}$ corresponds to fractions $\frac{a}{11c}$ and 0 to $\frac{a}{c}$ w/ $11 \nmid c$.

$\Rightarrow \mathrm{Hom}_{\Gamma_0(11)}(\Delta, \mathbb{C})$ is 2-dim.

$$\left.\begin{aligned} \varphi_0(x) &= \begin{cases} 1 & x \sim \infty \\ 0 & x \sim 0 \end{cases} \\ \varphi_1(x) &= \begin{cases} 0 & x \sim \infty \\ 1 & x \sim 0 \end{cases} \end{aligned}\right\} \text{these span the space.}$$

$$\varphi_0\big|_{\Delta_0} = -\varphi_1\big|_{\Delta_0}$$

So we get 1 new modular symbol.

$\Rightarrow \mathrm{Hom}_{\Gamma_0(11)}(\Delta_0, \mathbb{C})$ is at least 3-dimensional.

Are there any more?

$\Gamma = \Gamma_0(N)$. What info. determines a modular symbol?

- Δ_0 generated by $\{s\} - \{r\}$.
- collection of $\{s\} - \{r\}$ is generated by divisors of the form $\{\frac{b}{d}\} - \{\frac{a}{c}\}$ w/ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (
  (unimodular path)

<u>Example:</u> $\{\frac{11}{4}\} - \{\frac{3}{1}\} = \left(\{\frac{11}{4}\} - \{\frac{3}{1}\}\right) + \left(\{\frac{3}{1}\} - \{\frac{-2}{-1}\}\right)$
$+ \left(\{\frac{2}{1}\} - \{\frac{1}{0}\}\right) + \left(\{\frac{1}{0}\} - \{\frac{0}{1}\}\right).$

Given $\alpha \in SL_2(\mathbb{Z})$, $[\alpha] = \{\frac{b}{d}\} - \{\frac{a}{c}\}$.
$\overset{\shortparallel}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

- φ is determined by its values in $[\alpha]$ $\forall\, \alpha \in SL_2(\mathbb{Z})$

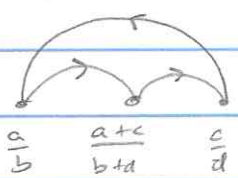
- Γ-inv. of $\varphi \Rightarrow \varphi([\alpha])$ only depends on α in $\Gamma_0(N) \backslash SL_2(\mathbb{Z})$

$$\Gamma_0(N) \backslash SL_2(\mathbb{Z}) \simeq \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto [c:d].$$

$N = 11$: mod symbols is at most 12 dimensional.

- Note $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = -\left[\begin{pmatrix} -b & a \\ -d & c \end{pmatrix}\right]$

$$\varphi([c;d]) = -\varphi([-d:c]).$$

- 

$\rightsquigarrow$ 3-term relation
"Manin relations"

Solve the relations and get the modular symbols for $\Gamma_0(11)$ are 3-dimensional.

Higher weights?

Need non-trivial coefficients.

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$V = \overset{\text{right}}{\mathbb{Z}[\Gamma]}$- module

$\varphi \in \mathrm{Hom}(\Delta_0, V)$ $\circlearrowleft$ right action of Γ $\qquad \gamma \in \Gamma$

$$(\varphi|\gamma)(D) := \varphi(\gamma D)|\gamma$$

Modular symbols with values in V and level Γ:

$$= \mathrm{Hom}_\Gamma(\Delta_0, V)$$

$$\varphi|\gamma = \varphi \Rightarrow \varphi(\gamma D) = \varphi(D)|\gamma^{-1}.$$

Take $V = V_k = \mathrm{Sym}^k(\mathbb{C}^2)$

$\uparrow$ homog. polys in X, Y of deg k.

$$(P|\gamma)(X,Y) = P((X,Y)\gamma^*) \qquad \gamma^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$$S_{k+2}(\Gamma) \longrightarrow \mathrm{Hom}_\Gamma(\Delta_0, V_k)$$

$$f \longmapsto \underbrace{\left(\{s\}-\{r\} \longmapsto 2\pi i \int_r^s (zX+Y)^k f(z)\,dz\right)}_{\Psi_f}.$$

- $\Psi_f(\gamma D) = \Psi_f(D)|\gamma^{-1}$
- $\Psi_f(0-\infty) = \sum c_i X^i Y^{k-i}$

$$c_i \longleftrightarrow L(f, i+1)$$

Hecke action:

$$\mathrm{Hom}_\Gamma(\Delta_0, V_k) \circlearrowleft T_n = \Gamma\begin{pmatrix}1&0\\0&n\end{pmatrix}\Gamma$$

e.g. $\Gamma = \Gamma_0(N)$ $\qquad (\varphi \mid T_\ell) = \sum_{a=0}^{\ell-1} \varphi \Big| \begin{pmatrix}1&a\\0&\ell\end{pmatrix} + \varphi \Big| \begin{pmatrix}\ell&0\\0&1\end{pmatrix}$

$\ell \nmid N$

(*) $f \rightsquigarrow \varphi_f$ preserves Hecke.

Eichler-Shimura: There is an isom. of Hecke-modules

$$M_{k+2}(\Gamma) \oplus S_{k+2}(\Gamma) \xrightarrow{\sim} \mathrm{Hom}_\Gamma(\Delta_0, V_k).$$

Remarks: 1) $M_{k+2}(\Gamma) \longleftrightarrow$ plus part

2) $S_{k+2}(\Gamma) \longleftrightarrow$ minus part

3) Eisenstein series $\longleftrightarrow$ "boundary symbols" on Δ.

4) RHS is completely algebraic and completely computable.

$V_k = \mathrm{Sym}^k(\mathbb{C}^2)$ can now be replaced by $\mathrm{Sym}^k(\mathbb{Q}^2)$ or $\mathrm{Sym}^k(\mathbb{Z}^2)$.

(in fact this gives eigenvalues are alg. #'s!)

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p-adic families:

Let $S_k(\Gamma_0(Np))^{ord}$ denote the space of p-ordinary forms.
(eigenvalue of U_p is a unit) $p \nmid N$.

<u>Fact:</u> $\dim(S_k(\Gamma_0(Np))^{ord})$ only depends on $k \bmod p-1$.

<u>Example:</u> $\underbrace{S_{2}(\Gamma_0(15))}_{1-\text{dim}}$ $\quad p=3$

$$q - q^2 - q^3 - q^4 + \cdots$$

The fact above then implies $\dim(S_k(\Gamma_0(15))^{ord}) = 1$
for all even k.
Let f_k denote the unique normalized ordinary form.
Moreover,

$$f_k \equiv f_{k'} \ (3^n) \quad \text{when} \quad k \equiv k' \ (3^{n-1}).$$

"Hida theory"
Hida interpolates the spaces $S_k(\Gamma_0(Np))^{ord}$ as k varies.

Clearly this cannot be true for all forms as
$\dim(S_k(\Gamma)) \to \infty$ as $k \to \infty$.

Coleman replaced $M_k(\Gamma_0)$ w/ $M^{\dagger}(\Gamma_0)$ = overconvergent modular forms.
$(\Gamma_0 = \Gamma_0(Np))$
∞-dim. space.

- $M_k(\Gamma_0) \hookrightarrow M_k^{\dagger}(\Gamma_0)$
- $f \in M_k^{\dagger}(\Gamma_0)$ with "small slope", then it is classical, i.e., $f \in M_k(\Gamma_0)$.

- $M_k^{\dagger}(\Gamma_0)$ interpolates p-adically..

Modular Symbols Analogue:

Recall $\mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k) \longleftrightarrow M_{k+2}(\Gamma_0)$.

Replace V_k w/ D_k some ∞-dim. space of distributions.

$D_k \twoheadrightarrow V_k$.

$\mathrm{Hom}_{\Gamma_0}(\Delta_0, D_k) \twoheadrightarrow \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k)$.

Distributions:

Let $\mathbb{A} = \{$ convergent power series over $\mathbb{Q}_p$ on the closed unit disc of $\mathbb{C}_p\}$.

$$= \left\{ \sum_{n=0}^{\infty} a_n z^n \;\middle|\; a_n \in \mathbb{Q}_p,\ |a_n|_p \to 0 \right\}$$

$\mathbb{A}$ is a Banach space:

$$\left\| \sum a_n z^n \right\| = \max_n |a_n|$$

$\mathbb{D} = \mathrm{Hom}_{cts}(\mathbb{A}, \mathbb{Q}_p)$ again a Banach space. $\quad \|\mu\| = \sup_{\substack{f \in \mathbb{A} \\ f \neq 0}} \frac{|\mu(f)|}{\|f\|}$

Notation: $\mu \in \mathbb{D}$, $f \in \mathbb{A}$

$$\mu(f) =: \int f \, d\mu$$

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Moments:

- the span of the monomials $\{z^j\}_{j=0}^{\infty}$ is dense in $\mathbb{A}$.

  $\Rightarrow$ $\mu \in \mathbb{D}$ is uniquely determined by the sequence $\{\mu(z^j)\}_{j=0}^{\infty}$

$$\Rightarrow \quad \mathbb{D} \hookrightarrow \prod_{j=0}^{\infty} \mathbb{Q}_p$$

$$\mu \longmapsto (\mu(z^j)).$$

In fact, the image is the bounded seqs. in $\mathbb{Q}_p$.

(Take any bounded seq. $\{\alpha_n\}$. Want $\mu(z^j) = \alpha_j$.
Define $\mu(\sum a_n z^n) = \sum a_n \alpha_n$.)

$\mathbb{D} \longleftrightarrow$ bounded seqs in $\mathbb{Q}_p$

$\mathbb{A} \longleftrightarrow$ seqs in $\mathbb{Q}_p$ tending to 0

Matrix actions:

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \ : \ p \nmid a,\ p \mid c \right\}.$$

Fix $k \geq 0$, $\gamma \in \Sigma_0(p)$, $f \in \mathbb{A}$.

$$(\gamma_k f)(z) := (a+cz)^k f\left(\frac{b+dz}{a+cz}\right)$$

$\mu \in \mathbb{D}$,

$$(\mu|_k \gamma)(f) := \mu(\gamma_k f).$$

Write $\mathbb{A}_k$ and $\mathbb{D}_k$ when considering the action via k.

We consider $\mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D}_k)$ to be the space of overconvergent modular symbols (oms) of weight k.

Specialization:

$$\rho_k : \mathbb{D}_k \longrightarrow V_k(\mathbb{Q}_p) = \mathrm{Sym}^k(\mathbb{Q}_p^2)$$

$$\mu \longmapsto \int (Y - zX)^k \, d\mu. = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mu(z^j) X^j Y^{k-j}$$

This is $\Sigma_0(p)$-equivariant.

Example: $k=0$ (so weight 2)

$$\rho_0 : \mathbb{D} \longrightarrow \mathbb{Q}_p$$

$$\mu \longmapsto \mu(1)$$

$$(\mu|\gamma)(1) = \mu(\gamma \cdot 1) = \mu(1) = \mu(1)|\gamma.$$

$$\rho_k^* : \mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D}) \longrightarrow \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k(\mathbb{Q}_p)).$$

specialization

Slopes of modular forms:

$f \in S_k(\Gamma_0)$ eigenform $f|U_p = \lambda f$

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slope of f = $\mathrm{ord}_p(\lambda)$ $\qquad$ $\mathrm{ord}_p(p) = 1$

$\uparrow$ p-adic valuation

<u>Fact:</u> slope $f \leq k-1$.

The reason: ① p-new $\Rightarrow a_p = \lambda = \pm p^{\frac{k}{2}-1}$

② p-old $\Rightarrow \exists\ g$ on $\Gamma_0(N)$ s.t. $f \in \mathrm{Span}(g(z), g(pz))$.

$\circlearrowleft U_p$

$$\mathrm{char}(U_p) = x^2 - a_p x + p^{k-1}$$

$\Rightarrow$ valuation of roots $\leq k-1$.

$h \in \mathbb{R}$ $\qquad$ $M^{(\leq h)}$ = subspace of M where U_p acts with slope $< h$

where M is some Hecke module.

<u>Thm (Stevens):</u> $\swarrow$ need to shift b/c of $k \leftrightarrow k+2$..

$$\mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D}_k)^{(<k+1)} \xrightarrow{\sim} \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k(\bar{\mathbb{Q}}_p))^{(<k+1)}$$

(Analogue of Coleman's "small slope" $\Rightarrow$ classical thm)

Let $f \in S_{k+2}(\Gamma_0)$ eigenform. The associated modular symbol is $\Psi_f^{\pm} \in \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k(\mathbb{C}))$

$$\rightsquigarrow \varphi_f^{\pm} = \Psi_f^{\pm} / \Omega_f^{\pm} \in \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k(\bar{\mathbb{Q}}_p)).$$

$$\varphi_f = \varphi_f^{+} + \varphi_f^{-}$$

Assume slope of $f < k+1$. By the control theorem there exists. a unique Φ_f Hecke eigensymbol lifting φ_f.

<u>Thm (Stevens):</u> $\Phi_f(0-\infty) = p$-adic L-function of f.

<u>p-adic L-functions:</u>

$$L_p(f) = \text{gadget "knows" } \frac{L(f,\overset{\text{cond } p^n}{\chi},1)}{\Omega_f^{\pm}} \in \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$$

Given χ,

$$L_p(f)(\chi) = c \cdot \frac{L(f,\chi,1)}{\Omega_f^{\pm}}$$

$L_p(f) =$ distribution, i.e., dual to some nice space of functions.

$L_p(f) \in \mathbb{D} = \mathrm{Hom}(\mathbb{A}, \mathbb{Q}_p)$

But $\chi \notin \mathbb{A}$!

$\mathcal{A}$ = locally analytic functions on $\mathbb{Z}_p$

$\mathbb{A} \hookrightarrow \mathcal{A}$

$\mathcal{D} = \mathrm{Hom}(\mathcal{A}, \mathbb{Q}_p) \qquad \mathcal{D} \hookrightarrow \mathbb{D}$

<u>Fact:</u> $\mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathcal{D})^{(<h)} \xrightarrow{\sim} \mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D})^{(<h)}$.

Recall:

$$\mathrm{Hom}_{\Gamma}(\Delta_0, V) =: \mathrm{Symb}_{\Gamma}(V) \to H^1(\Gamma, V)$$
$$\varphi \longmapsto (\gamma \mapsto \varphi(\gamma\infty - \infty))$$

$$\mathbb{A}_k = \{ \sum a_n z^n \mid |a_n| \to 0,\ a_n \in \mathbb{Q}_p \}$$
$$\mathbb{D}_k = \mathrm{Hom}_{cts}(\mathbb{A}, \mathbb{Q}_p). \qquad \mathbb{D}_k \twoheadrightarrow V_k(\mathbb{Q}_p)$$

Thm (Stevens):
$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)^{(<k+1)} \to \mathrm{Symb}_{\Gamma_0}(V_k)^{(<k+1)}$$
$$\text{IS}$$
$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D})^{(<k+1)}$$

Idea: $k=0$ slope $=0$

Fix φ a classical eigensymbol w/ U_p eigenvalue λ, $\mathrm{ord}_p(\lambda) = 0$.

Let Φ be any lifting of φ.

U_p is a compact operator, so its eigenvalues $\lambda_1, \lambda_2, \ldots$ with valuation $\to \infty$.

Imagine $\Phi = \Phi_1 + \Phi_2 + \ldots$ with Φ_i having eigenvalue λ_i

Assume further that $\mathrm{ord}_p(\lambda_i) > 0$ if $i > 1$.

Apply the operator U_p/λ again and again...

$$\underbrace{\frac{1}{\lambda^N} \Phi | U_p^N}_{\text{still lifts } \varphi} = \frac{\lambda_1^N}{\lambda^N} \Phi_1 + \frac{\lambda_2^N}{\lambda^N} \Phi_2 + \ldots$$

$$\rho^*\left(\frac{1}{\lambda^N}\Phi|U_p^N\right) = \frac{1}{\lambda^N}\rho^*(\Phi)|U_p^N$$
$$= \frac{1}{\lambda^N}\varphi|U_p^N = \varphi.$$

We have $\frac{\lambda_i^N}{\lambda^N}\Phi_i \to 0$ for $i>1$.

$\left\{\frac{1}{\lambda^N}\Phi|U_p^N\right\} \longrightarrow \Phi_1$. (This is a bogus arg., but gives an idea of what is going on.)

Make this more rigorous.

1) We need to justify that Φ exists.

2) $\left\{\frac{\Phi|U_p^N}{\lambda^N}\right\}$ prove is Cauchy converging to say $\tilde{\Phi}$

3) $\tilde{\Phi}$ is an eigensymbol lifting φ.
($\tilde{\Phi}$ is indep. of lift)

3) $\tilde{\Phi}$ lifts φ: this is true since each $\frac{\Phi|U_p^N}{\lambda^N}$ does.

$\tilde{\Phi}|U_p = \lambda\tilde{\Phi}$? :

$$\tilde{\Phi}|U_p = \left(\lim_{N\to\infty}\frac{1}{\lambda^N}\Phi|U_p^N\right)|U_p$$
$$= \lim_{N\to\infty}\left(\frac{1}{\lambda^N}\Phi|U_p^{N+1}\right)$$
$$= \lambda\lim_{N\to\infty}\left(\frac{1}{\lambda^{N+1}}\Phi|U_p^{N+1}\right)$$
$$= \lambda\tilde{\Phi}.$$

2) $\left\{ \frac{\Phi | U_p^N}{\lambda^N} \right\}$ is Cauchy: $N \le M$

$$\frac{\Phi | U_p^N}{\lambda^N} - \frac{\Phi | U_p^M}{\lambda^M} = \frac{1}{\lambda^N}\underbrace{\left(\Phi - \frac{\Phi | U_p^{M-N}}{\lambda^{M-N}} \right)}_{\text{in the kernel of specialization since both terms specialize to } \varphi.} \Big| U_p^N$$

<u>Claim:</u> $\Psi \in \ker(\rho^*) \Rightarrow \Psi | \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$ is divisible by p.

$\|\Psi\| = 1$

(Claim $\Rightarrow$ Cauchy)

<u>Subclaim:</u> $\mu \in \mathbb{D}$, $\mu(1) = 0$, $\|\mu\| = 1 \Rightarrow \mu | \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$ is divisible by p.

(subclaim $\Rightarrow$ claim)

<u>Pf:</u> $\left(\mu | \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}\right)(z^j) = \mu\left(\begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} z^j\right)$

$$= \mu\left((a+pz)^j\right)$$

$$= \mu\left(a^j + \binom{j}{1} a^{j-1} pz + \cdots + (pz)^j\right)$$

$$= \mu(p(\text{——})) \quad \text{using } \mu(a^j) = a^j \mu(1) = 0$$

4) Uniqueness of $\tilde{\Phi}$: Pick 2 lifts Φ and Φ' (of φ).

$\Phi - \Phi' \in \ker(\rho^*)$.

By previous claims $(\Phi - \Phi') | U_p^N \to 0$.

<u>Approximating distributions:</u>

$$\mu \in \mathbb{D} \longleftrightarrow \{\mu(z^j)\}_{j=0}^{\infty}$$

Fix $M, N >> 0$, Consider the first N moments of μ, each mod p^M.

Unfortunately, this data is not stable under the action of $\Sigma_0(p)$. (i.e., this data for μ does not determine the same data for $\mu|\gamma$)

Indeed, $\mu(z^j)=0 \quad 0 \le j \le N$, it is not true that $\mu|\gamma$ has the same property.

Let $\mu_4 \in \mathbb{D}$:

$$\mu_4(z^j) = \begin{cases} 1 & j=4 \\ 0 & o/w. \end{cases}$$

$\gamma = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}$

$$(\mu_4|\gamma)(1) = \mu_4(\gamma \cdot 1) = \mu_4(1) = 0$$

$$(\mu_4|\gamma)(z) = \mu_4(\gamma z) = \mu_4\left(\tfrac{z}{1-pz}\right) = \mu_4(z(1+pz+p^2z^2+\cdots)) = p^3$$

$$(\mu_4|\gamma)(z^2) = \mu_4(\gamma z^2) = \mu_4(z^2(1+pz+p^2z^2+\cdots)^2) = 3p^2$$

$$(\mu_4|\gamma)(z^3) = \cdots = 3p$$

$\mathbb{D}^0$ = unit ball of $\mathbb{D}$ $\quad (\mu(z^j) \in \mathbb{Z}_p)$

$\mathrm{Fil}^M(\mathbb{D}^0) = \{\mu \in \mathbb{D}^0 : \mu(z^j)$ is div. by $p^{M-j} \quad 0 \le j \le M\}$.

__Fact:__ $\mathrm{Fil}^M(\mathbb{D}^0)$ is stable under $\Sigma_0(p)$.

__Def:__ $\mathcal{F}(M) = \mathbb{D}^0 / \mathrm{Fil}^M(\mathbb{D}^0)$. ← finite approx. module

$\circlearrowleft_{\Sigma_0(p)}$

$$= \mathbb{Z}/p^M \times \mathbb{Z}/p^{M-1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p$$

$\mathbb{D}^\circ = \varprojlim \mathcal{F}(M)$.

Reprove the comparison thm (M. Greenberg):

Modify the filtration

$$\widetilde{\mathrm{Fil}}^M(\mathbb{D}^\circ) = \mathrm{Fil}^M(\mathbb{D}^\circ) \cap \ker \rho.$$

$$\widetilde{\mathcal{F}}(M) = \mathbb{D}^\circ / \widetilde{\mathrm{Fil}}^M(\mathbb{D}^\circ) \simeq \mathbb{Z}_p \times \mathbb{Z}/p^{M-1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$$

$$\begin{array}{c} \mathrm{Symb}_{\Gamma_0}(\mathbb{D}^\circ) \\ \downarrow \\ \varphi_M \in \mathrm{Symb}_{\Gamma_0}(\mathcal{F}(M)) \\ \downarrow \quad \downarrow \\ \varphi \in \mathrm{Symb}_{\Gamma_0}(\mathbb{Z}_p) \end{array}$$

Build φ_M taking values in $\mathcal{F}(M)$, $\varphi_M \to \varphi_{M-1}$,

$\varphi_M | U_p = \lambda \varphi_M$.

Assume we have such a φ_M and build φ_{M+1}.

$\varphi_M \in \mathrm{Hom}_{\Gamma_0}(\Delta_0, \widetilde{\mathcal{F}}(M))$.

Pick $\psi_{M+1} : \Delta_0 \longrightarrow \widetilde{\mathcal{F}}(M+1)$ lifting (as a map) φ_M. (any lift is fine)

$\psi_{M+1} \in \mathrm{Maps}(\Delta_0, \widetilde{\mathcal{F}}(M+1))$.

set maps

Magic: $\varphi_{M+1} = \psi_{M+1} | \frac{U_p}{\lambda}$.

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φ_{M+1} is additive, Γ_0-inv., indep. of lift and satisfies
$\varphi_{M+1}|U_p = \lambda \varphi_{M+1}$.

Why the magic? $\gamma_a = \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$

$$\varphi_{M+1}(D+D') - \varphi_{M+1}(D) - \varphi_{M+1}(D')$$

$$= \frac{1}{\lambda} \sum_{a=0}^{p-1} \underbrace{\left(\Psi_{M+1}(\gamma_a D + \gamma_a D') - \Psi_{M+1}(\gamma_a D) - \Psi_{M+1}(\gamma_a D')\right)}_{\widetilde{F}il^{M}(\mathbb{D}^{0})} \Big| \begin{pmatrix} p & a \\ 0 & p \end{pmatrix}$$

check that $\begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$ kills this