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~~Dasgupta~~
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Shintani zeta functions and Stark units 1:

Cases known for rank 1 abelian Stark conjecture:

TR_∞: $F=\mathbb{Q}$ example worked out

ATR: F imag. quadratic (elliptic units constructed using CM theory)

TR_p: $F=\mathbb{Q}$ Stickelberger's thm

Gross' refinement

other totally real base fields (under hypotheses) by

Darmon-Dasgupta-Pollack.

Only F s.t. Stark is known $\forall K/F$ is $F=\mathbb{Q}$ or $F=$ imag. quad.

Motivation:

F imag. quad., K/F abelian.

$S=\infty_F \cup \{\text{places of } F \text{ ramified in } K\}$.

$v=\infty_F$, $w|v$.

Stark: $\zeta'_{K/F,S}(\sigma,0)=\frac{1}{e}\log|u^\sigma|_w$

$v_1,v_2: F\hookrightarrow\mathbb{C}$ $\quad \bar{v}_1=v_2$

$w_1,w_2: K\hookrightarrow\mathbb{C}$ $\quad \bar{w}_1=w_2$

$$\zeta'_{K/F,S}(\sigma,0)=\frac{1}{e}\left[\log(w_1(u^\sigma))+\log(w_2(u^\sigma))\right].$$

CM theory: $\omega_i(u^\sigma) =$ CM value of an elliptic unit.

<u>Idea:</u>

$$\zeta_{K/F,S}(\sigma,s) = Z_1(s) + Z_2(s) \qquad [F:\mathbb{Q}]=2.$$

- $Z_i(s)$ are not well-defined
- $Z_i'(0)$ is mostly well-defined

* Decomposition arises in Shintani's proof of rationality of ζ-values of totally real fields at non-positive integers (1976).

<u>Shintani zeta functions:</u>

$a \in \mathrm{Mat}_{n\times d}(\mathbb{C})$, $\mathrm{Re}(a_i^j) > 0 \quad \forall i,j$

$x \in \mathbb{R}^d_{\geq 0}$, $x \neq 0$.

<u>Def:</u> $\zeta(a,x,s) = \sum_{k \in \mathbb{Z}^d_{\geq 0}} N(a(x+k))^{-s} \qquad \mathrm{Re}(s) > d/n.$

$$N\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1 \cdots c_n.$$

<u>Remark:</u> $n=1$ case: Hurwitz zeta functions.

<u>Thm:</u> $\zeta(a,x,s)$ admit mero. continuation to $\mathbb{C}$.
(proof later).

We now see the connection with partial zetas.

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Ex: $F/\mathbb{Q}$ real quadratic, $x \mapsto x_i$ embs.

$\mathfrak{f} \subset \mathcal{O}_F$ s.t. $E(\mathfrak{f}) = \langle \varepsilon \rangle$, $0 < \varepsilon < 1$

$K = K_{\mathfrak{f}}$ ray class field

$\mathfrak{a} \subset \mathcal{O}_F$, $(\mathfrak{a}, S_{\min}) = 1$, $(\mathfrak{a}, \mathfrak{f}) = 1$.

$$\zeta_{K/F,S}(\sigma_{\mathfrak{a}}, s) = \sum_{\substack{\mathfrak{b} \subseteq \mathcal{O}_F \\ (\mathfrak{b}, S) = 1 \\ \mathfrak{b} \sim \mathfrak{a}}} N\mathfrak{b}^{-s}$$

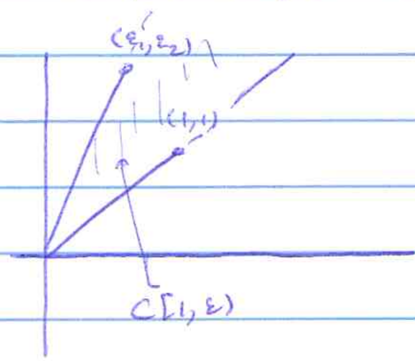
$$= N\mathfrak{a}^{-s} \sum_{z \in (1+\mathfrak{a}^{-1}\mathfrak{f})_{>0}/E(\mathfrak{f})} Nz^{-s}$$

Exercise: 1) $(1+\mathfrak{a}^{-1}\mathfrak{f})_{>0}/E(\mathfrak{f}) \xrightarrow{\sim} \{\mathfrak{b} \subseteq \mathcal{O}_F,\ (\mathfrak{b}, S) = 1,\ \mathfrak{b} \sim \mathfrak{a}\}$

$$x \longmapsto x\mathfrak{a}$$

2) $F_{>0} = \coprod_{n \in \mathbb{Z}} \varepsilon^n C[1, \varepsilon)$

$$C[1,\varepsilon) = \mathbb{Q}_{>0} 1 + \mathbb{Q}_{>0} \varepsilon$$



By 2) $(1+\mathfrak{a}^{-1}\mathfrak{f})_{>0} \cap C(1,\varepsilon) \xrightarrow{\sim} (1+\mathfrak{a}^{-1}\mathfrak{f})_{>0}/E(\mathfrak{f})$

$$\therefore \quad \zeta_{K/F,S}(\sigma_{\mathfrak{a}}, s) = N\mathfrak{a}^{-s} \sum_{z \in (1+\mathfrak{a}^{-1}\mathfrak{f}) \cap C[1,\varepsilon)} Nz^{-s}.$$

Asymmetric boundary conditions inconvenient. To fix this,

$$C[1,\varepsilon) = C(1) \cup C(1,\varepsilon).$$

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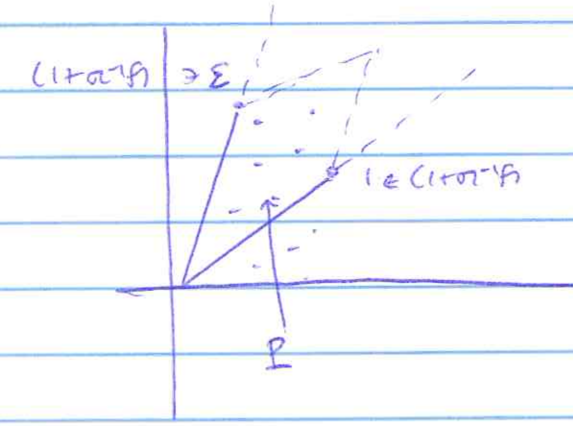
$$\zeta_{K/F,S}(\sigma_{\mathfrak{a}}, s) = N\mathfrak{a}^{-s}\left(\sum_{(1+\mathfrak{a}^{-1}\mathfrak{f})\cap C(1)} Nz^{-s} + \sum_{(1+\mathfrak{a}^{-1}\mathfrak{f})\cap C(1,\varepsilon)} Nz^{-s}\right).$$



$$(1+\mathfrak{a}^{-1}\mathfrak{f}) \cap C(1,\varepsilon) = \coprod_{\underbrace{z \in (1+\mathfrak{a}^{-1}\mathfrak{f}) \cap P}_{\text{finite}}} (z + \mathbb{Z}_{\geq 0} \cdot 1 + \mathbb{Z}_{\geq 0}\varepsilon).$$

$$\sum_{(1+\mathfrak{a}^{-1}\mathfrak{f})\cap C(1,\varepsilon)} Nz^{-s} = \sum_{\substack{z \in (1+\mathfrak{a}^{-1}\mathfrak{f})\cap C(1,\varepsilon) \\ z = x_1 + x_2\varepsilon \\ x_i \in (0,1] \cap \mathbb{Q}}} \sum_{k \in \mathbb{Z}_{\geq 0}^2} N((x_1+k_1) + (x_2+k_2)\varepsilon)^{-s}$$

$$= \sum_z \sum_k N\left(\underbrace{\begin{pmatrix} 1 & \varepsilon_1 \\ 1 & \varepsilon_2 \end{pmatrix}}_{\in M_{2,2}(\mathbb{R}_{>0})}\begin{pmatrix} x_1+k_1 \\ x_2+k_2 \end{pmatrix}\right)^{-s}$$

<u>Thm: (Shintani):</u> F totally real field, $\exists$ a finite set $\mathcal{C}$ of cones in $F_{>0}$ s.t.

$$F_{>0} = \coprod_{\varepsilon \in E(\mathfrak{f})} \coprod_{C \in \mathcal{C}} \varepsilon^{n} C.$$

<u>Cor:</u> $N\mathfrak{a}^{s}\, \zeta_{K/F,S}(\sigma_{\mathfrak{a}}, s)$ is a finite sum of Shintani zeta functions

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Analytic Continuation and Shintani's Decomposition:

Euler: $$\int_0^\infty e^{-nt} t^s \frac{dt}{t} = \Gamma(s) n^{-s}$$

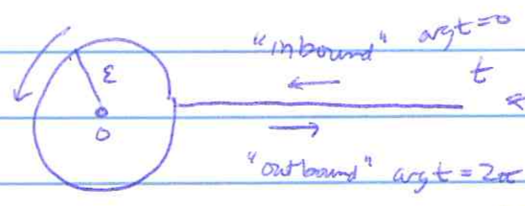
n - dim. version:

$$\Gamma(s)^n \zeta(a,x,s) = \int_{(0,\infty)^n} e^{-tax} \overbrace{\sum_{k \in \mathbb{Z}^d_{\geq 0}} e^{-tak}}^{\text{prod. of } d \text{ geom. series}} t^s \frac{dt}{t}$$

(where $t = (t_1, \ldots, t_n)$, $t^s \frac{dt}{t} = t_1^s \cdots t_n^s \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}$.)

$$= \int_{(0,\infty)^n} G(t)\, t^s \frac{dt}{t} \quad \text{where}$$

$$G(t) = \prod_{j=1}^d \left(\frac{e^{ta^j(1-x_j)}}{e^{ta^j} - 1} \right)$$



← integrate over this

this was used by Riemann.

$$I_\varepsilon = \int_{C(\infty,\varepsilon)^n} \prod_{j=1}^d \frac{e^{ta^j(1-x_j)}}{e^{ta^j} - 1} t^s \frac{dt}{t}$$

n = 1: (Hurwitz zeta)

denom. $e^{ta^j} - 1$ has isolated 0 at $t = 0$.

$n>1$ denom $= e^{t_1 a_1^j + \cdots + t_n a_n^j} - 1$

- zero along a hyperplane through 0

$C(\infty,\varepsilon)^n \supset$ polydisc of radius ε.

Integrand is not holo. on $C(\infty,\varepsilon)^n$ for any $\varepsilon>0$!

Shintani's fix:

$(0,\infty)^n = \coprod_{i=1}^{n} D_i$

$$D_i = \{ t : t_i \geq t_r \ \forall r=1,\ldots,n \}.$$

on D_i: $t = u(y_1,\ldots,y_{i-1},1,y_{i+1},\ldots,y_n)$

$\left(y_r = \frac{t_r}{u} \quad r \neq i \right)$

$y_r \in (0,1) \quad \forall\, r \neq i.$

$$\int_{(0,\infty)^n} = \int_{(0,\infty)} \int_{(0,1)^{n-1}}$$

$$J_\varepsilon = \int_{C(\infty,\varepsilon)} u^s \frac{du}{u} \int_{C(1,\varepsilon)} \prod_{j=1}^{d} \frac{e^{uya^j(1-x_j)}}{e^{uya^j}-1} \prod_{r\neq i} y_r^s \, dy_r .$$

$$e^{uya^j} - 1 = e^{u(y_1 a_1^j + \cdots + 1 \underset{>0}{a_i^j} + \cdots + y_n a_n^j)} - 1$$

$u \in C(\infty,\varepsilon) \Rightarrow |u| \geq \varepsilon.$

Consider $|y_r| = \varepsilon$

This contour integral is holomorphic on the contour for ε small enough.

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$$Z_i(a, x, s) = \frac{1}{\Gamma(s)^n \left(e^{2\pi n s} - 1\right)\left(e^{2\pi i s} - 1\right)^{n-1}} J_\varepsilon$$

$$\zeta(a, x, s) = \sum_{i=1}^{n} Z_i(a, x, s).$$

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Shintani zeta functions and Stark units II :

0) Recall from last time:

$a \in M_{n\times d}(\mathbb{C})$, $Re(a_i^j)>0$, $x \in \mathbb{R}^d_{\geq 0}$, $x \neq 0$

$$\zeta(a,x,s) = \sum_{k \in \mathbb{Z}^d_{\geq 0}} N(a(x+k))^{-s} \qquad Nv = v_1 \cdots v_n, \quad Re(s) > \frac{d}{n}$$

We showed

$$\zeta(a,x,s) = \sum_{i=1}^n Z_i(a,x,s)$$

where $Z_i(a,x,s) = C_n(s) \underbrace{\int_{C(\infty,\varepsilon)} \frac{du}{u} u^{-s} \int_{C(1,\varepsilon)^{n-1}} G(uy|_{y_i=1}) \prod_{r \neq i} y_r^i \frac{dy_r}{y_r}}_{I_\varepsilon(s)}$

$$C_n(s) = \frac{1}{\Gamma(s)^n (e^{2\pi i n s}-1)(e^{2\pi i s}-1)^{n-1}},$$

$$G(t) = \prod_{j=1}^n \frac{e^{t a^j (1-x_j)}}{e^{t a^j}-1} \qquad (\text{for } s \text{ small enough})$$

1) Values of $\zeta(a,x,s)$ at nonpos. integers:

Residue calculus gives

$$\underset{m \geq 1}{Z_i(a,x,1-m)} = \frac{(-1)^{n(m-1)}}{n} \frac{B_{i,m}(\mathbb{1}-x)}{m!} \qquad \mathbb{1} = (1,\dots,1)$$

$$\frac{B_{i,m}(\mathbb{1}-x)}{(m!)^n} = \text{coeff. of } (G(uy|_{y_i=1}), (u^a y_1 \cdots \hat{y}_i \cdots y_n)^{m-1})$$

Ex: value at $s=0$

$$I_\varepsilon(0) = \int_{C(\infty,\varepsilon)} \frac{du}{u} \int_{C(1,\varepsilon)^{n-1}} G(uy|_{y_i=1}) \frac{1}{y_1 \cdots \hat{y}_i \cdots y_r} dy_1 \ldots \hat{dy_i} \ldots dy_r$$

$$= (2\pi)^{n-1} \int_{C(\infty,\varepsilon)} \frac{du}{u} \prod_{j=1}^{d} \frac{e^{ua_i^j(1-x_j)}}{e^{ua_i^j}-1}$$

$$\prod_{j=1}^{d} = \prod_{j=1}^{d} \frac{1}{ua_i^j} \frac{(ua_i^j)e^{ua_i^j(1-x_j)}}{e^{ua_i^j}-1}$$

$\vdots$

$$Z_i(a,x,0) = \frac{(-1)^d}{n} \sum_{\substack{l_1+\cdots+l_d=d \\ (l_1,\ldots,l_d)}} \prod_{j=1}^{d} \frac{B_{l_j}(x_j)}{l_j!} (a_i^j)^{l_j-1}.$$

$$Z_i(a,x,0) \in \mathbb{Q}(\{x_i\}, \{a_i^j\}).$$

<u>Thm (Klingen-Siegel):</u> $\zeta_{K/F}(\sigma_{\mathfrak{f}}, 1-m) \in \mathbb{Q}$

<u>Pf (Shintani):</u> $a^1, \ldots, a^d \in F_{>0}$

$$\zeta(a,x,0) = \sum_{l} \underbrace{\mathrm{Tr}_{F/\mathbb{Q}}\left(\prod_{j=1}^{d} \left\{\frac{B_{l_j}(x_j)}{l_j!}(a^j)^{l_j-1}\right)}_{\text{Dedekind sums}} \in \mathbb{Q}$$

$(x_j \in \mathbb{Q})$

2) Shintani zeta functions for non-TR fields

$F = \#$ field, $v_i : F \hookrightarrow \mathbb{R}$ $i = 1, \ldots, r_1$,

$v_i : F \hookrightarrow \mathbb{C} \qquad i = r_1+1, \dots, n$

$f \subset \mathcal{O}_F \;, \quad K = K_f$

$\exists\ \mathcal{C}$ cones s.t.

$$F_{>0} = \coprod_{\varepsilon \in E(f)} \coprod_{c \in \mathcal{C}} \varepsilon c$$

$$c = \sum_{j=1}^{d(c)} \mathbb{Q}_{>0}\, a_c^j \qquad a^i \in F_{>0}$$

<u>Problem:</u> No guarantee $\mathrm{Re}(a_{c,i}^j) > 0$

<u>Fix:</u> At cost of refining $\mathcal{C}$ $\exists\ u_{c,i} \in \mathbb{C}$, $c \in \mathcal{C}$, $i = 1, \dots, n$ s.t.

① $|u_{c,i}| = 1 \quad \forall i$

② $v_i = \overline{v_i} \Rightarrow u_{c,i} = \overline{u_{c,i}}$

③ $\mathrm{Re}(u_{c,i}\, a_{c,i}^j) > 0$

$$u_c = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & u_{c,r_1+1} & \\ & & & & \ddots & \\ & & & & & u_{c,n} \end{pmatrix}$$

$$② \Rightarrow N(u_i\, a_c(x+k)) = N(a_c(x+k))$$

$$\Rightarrow \zeta_{K/F,S}(\sigma_{\mathfrak{a}}, s) = \sum_{c \in \mathcal{C}} \sum_{\substack{z \in (1+\mathfrak{a}^{-1} f) \cap c \\ z = \sum x_j a^j \\ x_j \in (0,1]}} \zeta(u_c a_c, x, s).$$

Define:

$$Z_i(\sigma_{\mathfrak{a}}, \mathcal{C}, \{u_i\}, s) = \sum_{c \in \mathcal{C}} \sum_{z \dots} Z_i(u_c a_c, x, a)$$

$$\zeta_{K/F}(\sigma_{\mathfrak{a}}, s) = N\mathfrak{a}^{-s} \sum_{i=1}^{n} Z_i(\sigma_{\mathfrak{a}}, \mathcal{C}, \{u_i\}, s)$$

Ex: F imag. quad.

$$\zeta'_{K/F}(\sigma_{\mathfrak{a}}\sigma_{\mathfrak{a}}, 0) = -\log|u^\sigma|_w \quad \forall \sigma \in \mathrm{Gal}(K/F)$$

$$= -\log^w(u^\sigma) - \log^w(\overline{u^\sigma})$$

$$Z_1'(0) + Z_2'(0)$$

Thm (Shintani): $Z_1'(-, 0) = \log(\text{cm-value of Siegel unit})$
(We'll see why this is indep. of choices we have made)

$n>2$ $\quad Z_i'(\sigma, \mathcal{C}, \{u_i\}, 0)$ is not independent of the choices we have made.

3) Dependence on choices:

Thm: ① $Z_i(\sigma, \mathcal{C}, \{u_i\}, 0)$ is well-defined

② If v_i is real, $Z_i'(\sigma, \mathcal{C}, \{u_i\}, 0)$ is well-defined.

③ Suppose v_i is complex. $N(c, x) = \text{denom.}(\zeta(u_c a_c, x, 0))$
~~Then~~ set

$$\varphi_i(a_c, x) := Z_i'(u_c a_c, x, 0) - \zeta(u_c a_c, x, 0)\log(u_{c,i}).$$

Then

$$\varphi_i(a_c, x) \bmod \frac{2\pi i \mathbb{Z}}{N(c, x)} \quad \text{is ~~well-defined~~ independent of } \{u_c\}$$

④ $\Phi_i(\sigma, \mathcal{C}) = \sum_{c \in \mathcal{C}} \sum_{z \in \cdots} \varphi_i(a_c, x)$

$$N(\sigma, \mathcal{C}) = \mathrm{lcm}(\{N(c, x)\})$$

$$\Phi_i(\sigma, \mathcal{C}) \bmod \frac{2\pi i}{N(\sigma, \mathcal{C})}\mathbb{Z} \quad \text{is indep of } \{u_c\}.$$

$$\sum_{i \in \eta}^{\wedge} \Phi_i(\sigma, \mathcal{C}) = \zeta'_{K/F,S}(\sigma_\alpha, 0).$$

<u>Worthwhile project:</u> "T-smoothing" this whole discussion, relate to Stark-Tate

$$\zeta'_{K/F,S,T}(0) = -\log|u_T^\sigma|_w$$

(u_T unique if it exists).

<u>Smoothing Shintani:</u>

- P: Cassou-Noguès const$^{\text{n}}$ of p-adic zeta function of totally real fields
- Dasgupta's refinement conjecture.

From now on we'll ignore all N's.

<u>Thm:</u> V_i complex, $\mathcal{C}, \mathcal{C}'$, $\exists\ \varepsilon \in E(\mathfrak{f})$ s.t.

$$\Phi_i(\sigma, \mathcal{C}') = \Phi_i(\sigma, \mathcal{C}) + \log \varepsilon_i \pmod{2\pi i \mathbb{Z}}.$$

<u>Pf:</u> • $\Phi_i(\sigma, \mathcal{C})$ is invariant under subdivision of $\mathcal{C}$ (Sczech)

• It suffices to consider:

$$\forall c \in \mathcal{C}\ \exists\ \varepsilon_c \in E(\mathfrak{f}) \text{ s.t.}$$

$$\mathcal{C}' = \{\varepsilon_c c : c \in \mathcal{C}\}$$

?

Then $a_{\varepsilon_c c} = \delta_c a_c$ $\quad \delta_c = \mathrm{diag}(\varepsilon \ldots \ldots)$

can take $U_{\varepsilon_c c} = u_c \delta_c^{-1}$

$$\varphi(a_{\varepsilon_i c}, x) - \zeta(u_{\varepsilon_i} c a_c, x, 0) \log u_{c,i} \cdot \varepsilon_{c,i}^{-1}$$

$$\equiv \varphi(a_c, x) - \zeta(u_c a_c, x, 0) \log(u_{c,i})$$

$\Rightarrow$ ~~φ(a...)~~

$$\varphi(a_{\varepsilon_c} c, x) \equiv \varphi(a_c, x) + \log \varepsilon_{c,i}^{\zeta(u_c a_c, x, 0)}$$

Now sum over $x, c, \ldots$ $\blacksquare$

4) Complex cubic fields (Ren - Sczech):

F complex cubic, v_1 real emb. $\overline{v_2} = v_3$.
$E(f) = \langle \eta \rangle$.

Want: $\vartheta \bmod 2\pi i \mathbb{Z}$ some comb. of $\Phi_1(\sigma, \tau)$ & $\Phi_2(\sigma, \tau)$
s.t.

① $\vartheta \bmod 2\pi i \mathbb{Z}$ is well-defined
② $(\vartheta + \bar{\vartheta}) = \zeta'_{K/F,S}(\sigma_{\mathfrak{a}}, 0)$.

Conj: $\vartheta = \log(\text{Stark unit})$.

$$\vartheta := \phi_1(\tau, \sigma) \frac{\log \eta_2}{\log \eta_1} - \phi_2(\tau, \sigma).$$

~~ϑ~~ $=$ ~~$\dfrac{\phi_1 \log \eta_2 + \log \eta_3 \ \phi_2}{\log \eta_1}$~~