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The Rank One Abelian Stark Conjecture:

Stark's conjectures relate values of derivatives of partial zeta functions at $s=0$

to

logarithms of absolute values of units in algebraic number fields.

- analogous to (and a refinement of) the Dirichlet class number formula.

- A version for units of the BSD conjecture for elliptic curves. (in fact, both can be simultaneously generalized and refined in the Equivariant Tamagawa Number Conjecture)

- "Abelian" indicates we will consider extensions K/F that are Galois with abelian Galois groups.

- "Rank one" means we'll consider 1^{st} derivatives of ζ-functions. (conjectures very precise here)

- Rubin later made a similar precise conjecture in the higher rank case.

Example: $f \in \mathbb{Z}_{\geq 2}$, $a \in \mathbb{Z}$ $(a,f)=1$.

Define $$\zeta_f(a,s) = \sum_{\substack{n=1 \\ n \equiv a \pmod f}}^{\infty} \frac{1}{n^s}. \qquad s \in \mathbb{C}, \ \mathrm{Re}(s) > 1$$

This is essentially ~~the~~ a Hurwitz zeta function.

$$\zeta_{Hur}(x,s) = \sum_{n\geq 0} \frac{1}{(x+n)^s} \qquad x,s\in\mathbb{C} \quad Re(x)\geq 0,\ Re(s)>1.$$

$$\zeta_f(a,s) = f^{-s}\,\zeta_{Hur}\left(\frac{a}{f},s\right) \qquad (0<a<f).$$

$$\zeta_f(a,s) = \frac{1}{f}\frac{1}{s-1} + \underset{?}{\overset{\shortparallel}{b(a,f)}} + (\quad)(s-1) + \cdots$$

Stark's conjectures concern $b(a,f)$.

Using the functional equation we can move from $s=1$ to $s=0$.

Define $\zeta_f^+(a,s) = \zeta_f(a,s) + \zeta_f(-a,s)$.

For $0<a<f$, we have

$$\zeta_f(a,0) = \frac{1}{2} - \frac{a}{f}.$$

$$\Rightarrow \zeta_f^+(a,0) = 0.$$

So

$$\zeta_f^+(a,s) = c(a,f)s + \cdots$$

$$\frac{d}{ds}\zeta_H(x,s)\Big|_{s=0} = \log\Gamma(x) - \frac{1}{2}\log(2\pi)$$

$$\Rightarrow c(a,f) = \log\left(\frac{\Gamma(\frac{a}{f})\,\Gamma(1-\frac{a}{f})}{2\pi}\right)$$

$$= -\log\left(2\sin\left(\frac{\pi a}{f}\right)\right)$$

$$= -\frac{1}{2}\log\left(2-2\cos\left(\frac{2\pi a}{f}\right)\right)$$

$$= -\tfrac{1}{2}\log(u(a,f)).$$

$u(a,f) = (1-\zeta_f)(1-\zeta_f^{-1})$ where $\zeta_f = e^{2\pi i \cdot a/f}$

Thus, we have $u(a,f) \in K = \mathbb{Q}(\zeta_f)^+ = \mathbb{Q}(\zeta_f + \zeta_f^{-1})$.

Exercise: $u(a,f) \in \mathcal{O}_K[\frac{1}{f}]^\times$

In fact, $u(a,f) \in \mathcal{O}_K^\times$ if f is divisible by at least 2 distinct primes.

Summary: $\zeta_f^+(a,0) = 0$

and

$$(\zeta_f^+)'(a,0) = -\frac{1}{2}\log u(a,f)$$

where

$$u(a,f) \in \begin{cases} \mathcal{O}_K^\times & f \text{ div. by at least 2 primes} \\ \mathcal{O}_K[\frac{1}{f}]^\times & f \text{ prime power} \end{cases}$$

where $K = \mathbb{Q}(\zeta_f)^+$.

The Conjecture:

K/F an abelian extension of number fields.

S = finite set of places of F
$\supseteq \{v \mid \infty\} \cup \{$ ~~places~~ finite places that ramify in $K\}$.

Assume $|S| \geq 3$. ($|S| = 2$ is ok, see notes)

$$\zeta_{K/F,S}(\sigma, s) = \sum_{\substack{\mathfrak{n} \subseteq \mathcal{O}_F \\ (\mathfrak{n}, S) = 1,\ \sigma_{\mathfrak{n}} = \sigma}} (N\mathfrak{n})^{-s} \qquad (s \in \mathbb{C},\ \mathrm{Re}(s) > 1).$$

$\sigma \in \mathrm{Gal}(K/F)$ elt of Galois assoc. to $\mathfrak{n}$ via CFT.

(When $F = \mathbb{Q}$, $K = \mathbb{Q}(\zeta_f)^+$, $\sigma = \sigma_a : \zeta_f \mapsto \zeta_f^a$
$S = \{\infty, p \mid f\}$, $\zeta_{K/F}(\sigma, s) = \zeta_f^+(a, s)$)

Let $U_v = U_v(K) = \{u \in K^\times : |u|_w = 1 \ \forall w \nmid v\}$.
where v is a place of F.
"strong v-units" since we allow w to be arch.

Fix $v \in S$. Assume v splits completely in K. Let w be a place of K above v.
v splits in $K \Rightarrow \zeta_{K/F,S}(\sigma, 0) = 0$

<u>Conjecture:</u> $\exists\ u \in U_v(K)$ s.t.
$$\zeta'_{K/F}(\sigma, 0) = -\frac{1}{e} \log |u^\sigma|_w \quad \forall \sigma \in \mathrm{Gal}(K/F)$$
$e = \#\mu(K)$. where u depends on w. Furthermore, $K(u^{1/e})/F$ is abelian.

<u>Note:</u> $|u|_{w'}$ is specified $\forall w'$ of K.

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$T = \{\mathfrak{C}\}$ prime ideal $\mathfrak{C} \subseteq \mathcal{O}_F$, $\mathfrak{C} \notin S$

$\mathrm{Char}(\mathcal{O}_F/\mathfrak{C}) = \ell \geq [F:\mathbb{Q}]+2.$

$$\zeta_{K/F,S,T}(\sigma,s) = \zeta_{K/F,S}(\sigma,s) - N\mathfrak{C}^{1-s}\,\zeta_{K/F,S}(\sigma\sigma_{\mathfrak{C}}^{-1},s)$$

Equivalent Conj.: $\exists\ u_T \in U_{v,T} = \{u \in U_v : u \equiv 1 \pmod{\mathfrak{C}\mathcal{O}_K}\}$

s.t.

$$\zeta'_{K/F,S,T}(\sigma,0) = -\log|u_T^{\sigma}|_w$$

u, u_T related: $\lambda = u^{1/e}$, $u_T = \lambda/\sigma_{\mathfrak{C}}^{-1}(\lambda)^{N\mathfrak{C}}$

u_T unique.

If S contains v, v' that split completely in K, then $\zeta_{K/F}(\sigma,0)=0$. $\Rightarrow u=1$ works in conjecture. So we only consider

- case TR_∞: F = totally real, v = real, places of K above v are real, all other arch. places are complex.
- case ATR: F = "almost totally real", i.e. F has exactly 1 complex place v. K = totally complex.
- case ~~TRR~~ TR_p: F totally real, v = finite place, K = totally complex.

Case TR_∞:

$$u^{\sigma} = e^{-2\zeta'_{K/F,S}(\sigma,0)}$$ by inverting the formula

This can be viewed as progress towards Hilbert's

12th problem

Case ATR:

$$|u_T^\sigma|_w = e^{-\zeta'_{K/F,S,T}(\sigma,0)}$$

But we don't see what u_T^σ is in $\mathbb{C}$, only its absolute value. What is $\arg(u_T^\sigma)$?

Motivating Question: Can we give an exact formula for $u_T^\sigma \in K_w$ and not just its absolute value?

ATR: Ren-~~[illegible]~~ Sczech, Charollois-Darmon yes!

TRp: Darmon-Dasgupta
Chapdelaine
Dasgupta
Charollois-Dasgupta
} Yes!

Idea: ζ-functions are not the whole story on the analytic side. ζ-functions are merely shadows of "bigger" or "more refined" objects (cohomology classes / Shintani zeta-functions)

Main obstruction: units in ground field F.

Case TRp: $v = \wp \subseteq \mathcal{O}_F$, prime ideal.

Let $R = S - \{\wp\}$, $w = \mathfrak{P} | \wp$.

$$\zeta_{K/F,S,T}(\sigma,s) = (1 - N\wp^{-s})\,\zeta_{K/F,R,T}(\sigma,s)$$

$$\zeta'_{K/F,S,T}(\sigma,0) = (\log N\wp)\,\zeta_{K/F,R,T}(\sigma,0).$$

$$-\log|u_T^\sigma|_{\mathfrak{P}} = (\log N\wp)\underbrace{\mathrm{ord}_{\mathfrak{P}}(u_T^\sigma)}_{\in\mathbb{Z}.}$$

Stark's conjecture in this case says:

$$\exists\ u_T \in U_{\wp,T} \text{ s.t. } \mathrm{ord}_{\mathfrak{P}}(u_T^\sigma) = \zeta_{K/F,R,T}(\sigma,0).$$

<u>Note:</u> RHS is in $\mathbb{Z}$ by work of Deligne-Ribet / Pi Cassou-Nogues / Barsky.