Computing Chow-Heegner Points: (to diagonal cycles)

$$V = X \times X \times X = X_1 \times X_2 \times X_3$$

$$X_1 \times X_2 \times X_3 \longrightarrow X \quad \left(\begin{array}{c} X_{12} \times X_{34} \subseteq X_1 \times X_2 \times X_3 \times X_4 \\ X_{12} = \text{diagonal in } X_1 \times X_2, \text{ likewise for } X_{34}. \end{array} \right)$$

$$\begin{tikzcd} \Phi : CH^2(X^3)_0 \arrow[r] \arrow[dr, bend right, "\Phi_E"'] & \mathrm{Jac}(X) \arrow[d, "\varphi_E"] \\ & E \end{tikzcd}$$

~~Given $T \in CH^2(X_1 \times X_2 \times X_3)_0$~~

Given $T \in \mathrm{Pic}(X_1 \times X_2) \longrightarrow \Delta_T \in CH^2(X^3)_0$

<u>Goal:</u> Compute $\Phi(\Delta_T)$

<u>Key Remark:</u> $\Phi(\Delta_T) = \pi_4 \left(\pi_{123}^{-1}(\Delta_T) \cdot (X_{12} \times X_{34}) \right)$.

$$\begin{tikzcd} & X_1 \times \cdots \times X_4 \arrow[dl, "\pi_{123}"'] \arrow[dr, "\pi_4"] & \\ X_1 \times X_2 \times X_3 & & X_4 \end{tikzcd}$$

This is not useful for calculations.

Recall the following analytic formula:

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$$\begin{tikzcd} CH^2(X^3)_0 \arrow[r, "AJ"] \arrow[d, "\Phi"] & \mathrm{Fil}^2 H^3_{dR}(X^3)^\vee / H_3(X^3(\mathbb{C}), \mathbb{Z}). \arrow[d] \\ \mathrm{Jac}(X) \arrow[r, "AJ"] \arrow[d, "\varphi_E"] & \Omega^1(X)^\vee / H_1(X(\mathbb{C}), \mathbb{Z}) \arrow[d, "ev_{\omega_E}"] \\ E \arrow[r] & \mathbb{C}/\Lambda_E \end{tikzcd}$$

$$\Phi_E(\Delta_T) = AJ(\Delta_T)(cl(\Delta_{12}) \otimes \omega_E) \qquad (*)$$

- $cl(\Delta_{12}) \in \mathrm{Fil}^1 H^2_{dR}(X_1 \times X_2) \cap H^2_B(X_1 \times X_2, \mathbb{Z})$
- $\omega_E \in \mathrm{Fil}^1 H^1_{dR}(X)$

A variant of $(*)$:

$$\Delta_{GKS} = \Delta_{123} - \Delta_{13} - \Delta_{12} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3$$

$$\Delta_{12} = \{(x, x, P)\}$$
$$\Delta_1 = \{(x, P, P)\}.$$

Then $\Delta_{GKS} \in CH^2(X^3)_0$.

$$AJ(\Delta_T)(cl(\Delta_{12}) \otimes \omega_E) = AJ(\Delta_{GKS})(cl(T) \otimes \omega_E). \qquad (**)$$

<u>Pf. of $(**)$:</u> Formal calculation.

$$cl(T) \in H^2_{dR}(X_1 \times X_2) = H^2_{dR}(X_1) \otimes H^0_{dR}(X_2) \oplus H^1_{dR}(X_1) \otimes H^1_{dR}(X_2) \oplus H^0_{dR}(X_1) \otimes H^2_{dR}(X_2).$$

Assume wlog, $cl(T) \in H^1_{dR}(X_1) \otimes H^1_{dR}(X_2)$.

Formula for $AJ(\Delta_{GKS})(\omega \otimes \eta \otimes \omega_E)$.

<u>Iterated integrals:</u> (Chen, Hain, ...)

Given $\gamma : [0,1] \longrightarrow X(\mathbb{C})$

ω, η closed smooth 1-forms on $X(\mathbb{C})$

$$\int_\gamma \omega \cdot \eta := \int_{0 \le t_2 \le t_1 \le 1} \gamma^* \omega(t_1)\, \gamma^* \omega(t_2)$$

$$= \int_\gamma \omega\, F_\eta$$

F_η = primitive of η on $\tilde{X}$ = universal covering space of $X(\mathbb{C})$.

$$F_\eta(\xi) = \int_{*}^{\xi} \eta.$$

Simplifying assumption: $\langle \omega, \eta \rangle = 0$

$\Rightarrow (\omega \wedge \eta)|_{X_{12}} = d\alpha$, α is of type $(1,0)$.

<u>Thm:</u> $AJ(\Delta_{GKS})(\omega \otimes \eta \otimes \omega_E) = \int_{\gamma_E} \omega \cdot \eta - \int_{\gamma_E} \alpha$,

$\gamma_E \in H_1(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C}$ is the Poincaré dual to ω_E.

$$\int_{\gamma_E} \xi = \langle \omega_E, \xi \rangle$$

<u>Proof:</u> Later. (if there is time)

<u>Problem:</u> α is not easy to calculate in practice

<u>Exception:</u> If $cl(T) = \omega \otimes \eta + \eta \otimes \omega$, then
$cl(T)|_{X_{12}} = \omega \wedge \eta + \eta \wedge \omega = 0$. In this case we can take $\alpha = 0$. In this case the formula gives:

$$AJ(\Delta_{GKS})((\omega \otimes \eta + \eta \otimes \omega) \otimes \omega_E) = \int_{\gamma_E} \omega \cdot \eta + \eta \cdot \omega$$

$$= \left(\int_{\gamma_E} \omega\right)\left(\int_{\gamma_E} \eta\right).$$

Because we can't get our hands on α, it is better to work with $H^1_{dR}(X) = \Omega^1_{\mathrm{II}}(X) / d\mathrm{II}(X)_0$ (differential forms of second kind)

<u>Notations:</u> X = projective curve
$Y = X - \{\infty\}$.

Assume wlog $\omega \in \Omega^1(X)$, η meromorphic differential of second kind on X.

F_η = meromorphic primitive of η on $\tilde{X}$.

$F_\eta(\sigma) = \int_\sigma \eta$ is well-defined.

ωF_η = mero. diff. on $\tilde{X}$.

Principal parts: $\alpha \in \Omega^1_{mer}(\tilde{X})$

$\underline{PP}_x(\alpha) \doteq$

$$\alpha_x = \sum_{j=-M}^{\infty} a_j \, t_x^j \cdot dt_x.$$

$$\underline{PP}_x(\alpha) = \left(\sum_{j=-M}^{-1} a_j \, t_x^j \right) dt_x$$

__Lemma:__ For all $x \in \tilde{X}$, $\underline{PP}_x(\omega F_\eta)$ depends only on image of x in $X(\mathbb{C})$.

It makes sense to write $\underline{PP}_x(\omega F_\eta)$, $x \in X(\mathbb{C})$.

__Prop.:__ There exists $\alpha \in \Omega^1_{mer}(X)$ s.t.

1) $\underline{PP}_x(\alpha) = \underline{PP}_x(\omega F_\eta) \quad \forall x \in Y$.
2) $\underline{PP}_\infty(\alpha) = \underline{PP}_\infty(\omega F_\eta) \quad (\text{mod } \frac{dq}{q})$

α is well-defined mod $\Omega^1(X)$

__Thm:__ $AJ(\Delta_{GKS})(\omega \otimes \eta \otimes \omega_E) = \int_{\gamma_E} \omega F_\eta - \int_{\gamma_E} \alpha$

$$= \int_{\gamma_E} \omega \otimes \eta - \int_{\gamma_E} \alpha.$$

__Algorithm for Chow-Heegner Points:__

__Set-up:__ $\cdot\ X = X_0(N)$
$\cdot\ Y = X_0(N) - \{\infty\}$

- $\varphi_E : X_0(N) \longrightarrow E. \longleftrightarrow f \in S_2(\Gamma_0(N))$
- $cl(T) = \sum \omega_i \otimes \eta_i$

(Ex: g = eigenform of wt 2 on $\Gamma_0(N)$ with rational coefficients, $g \neq f$.)

- $\omega_g \in \Omega^1(X)^g$
- $\eta_g \in H^1_{dR}(X)^g - \mathbb{C}\omega_g$

Fact: $\omega_g \otimes \eta_g - \eta_g \otimes \omega_g$ is a Hodge class in $H^2_{dR}(X_1 \times X_2)$
i.e., it belongs to $Fil^1 H^2_{dR}(X_1 \times X_2) \cap H^2_B(X_1 \times X_2, \mathbb{Z})$.

Strategy:

- $P(f,g) = AJ(\Delta_{GKS})((\omega_g \otimes \eta_g - \eta_g \otimes \omega_g) \otimes \omega_f)$
$\in \mathbb{C}/\Lambda_E = E(\mathbb{C})$.

Step 1: Compute q-exp. for ω_f, ω_g.
$f = \sum_{n=1}^{\infty} a_n(f) q^n, \quad g = \sum_{n=1}^{\infty} a_n(g) q^n.$

Step 2: Compute q-exp. of η_g.
One approach: - take $\omega_1, \ldots, \omega_t$ basis of $\Omega^1(X) = S_2(\Gamma_0(N))$.
- Let $u \in \mathcal{O}_Y$ with $ord_\infty(u)$ as small as possible.
$u\omega_1, \ldots, u\omega_t$.

"Claim:" $(\omega_1, \ldots, \omega_t, u\omega_1, \ldots, u\omega_t)$ is a basis for $H^1_{dR}(X)$

$\leadsto$ get η_g using modular symbols.

Step 3: Find a mero. diff. α on X, regular on Y s.t.

$$\underline{PP}_\infty(\alpha) = \underline{PP}_\infty(2\omega_g F_{\eta_g}) \quad (\text{mod } \tfrac{dq}{q}).$$

Step 4: Compute $\gamma_E \in H_1(X_0(N), \mathbb{C})^f$
(modular symbol algorithm)

Step 5: $$P(f,g) = 2\int_{\gamma_E} \omega_g \cdot \eta_g - \int_{\gamma_E} \alpha.$$